CS103
FALL 2025

Lecture 11:

Graph Theory

Part 3 of 3

Agenda for Today

- ***The Pigeonhole Principle***
  - A simple yet surprisingly effective fact.
- ***Graph Theory Party Tricks***
  - Cool tricks to try at your next group meeting.
- ***A Little Movie Puzzle***
  - Who watched what?

Recap from Last Time

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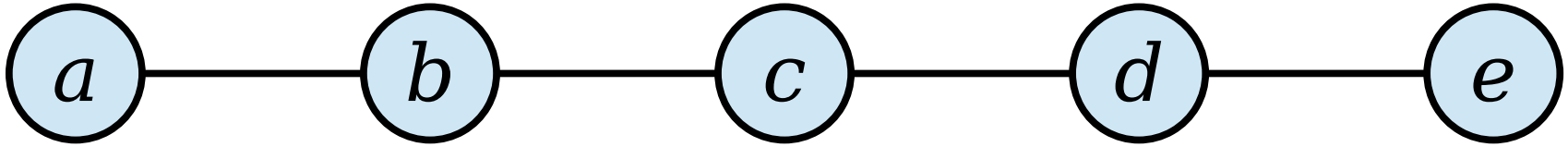
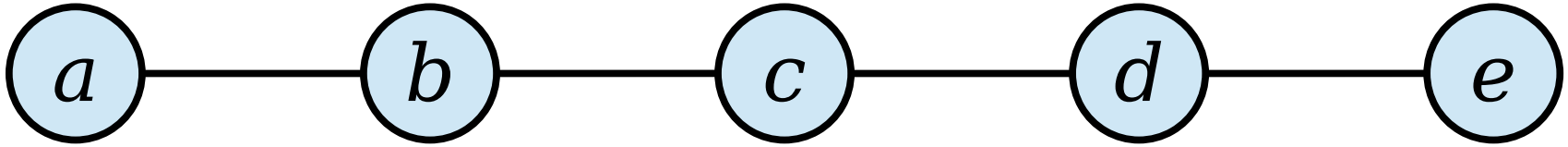
- When there's an edge between two nodes, we say they are _____.
- If there's a path between two nodes, we say they are _____ from one another.



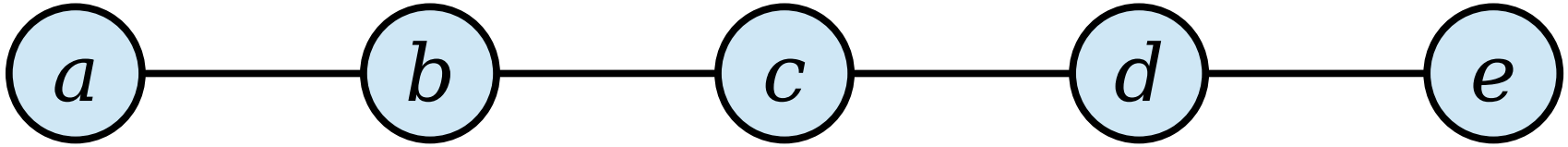
Recap from Last Time

- When there's an edge between two nodes, we say they are ***adjacent***.
- If there's a path between two nodes, we say they are _____ from one another.



Recap from Last Time

- When there's an edge between two nodes, we say they are ***adjacent***.
- If there's a path between two nodes, we say they are ***reachable*** from one another.



New Stuff!

The Pigeonhole Principle

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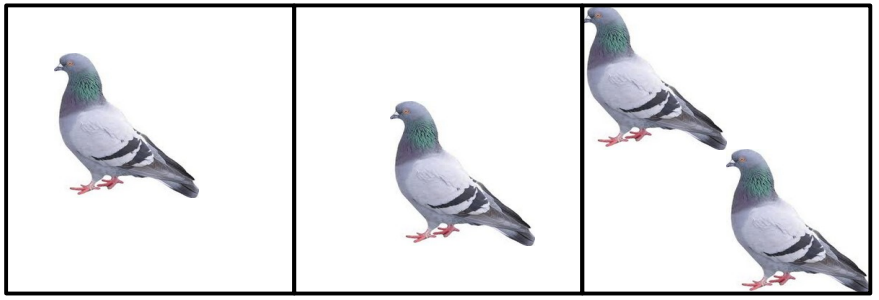
Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



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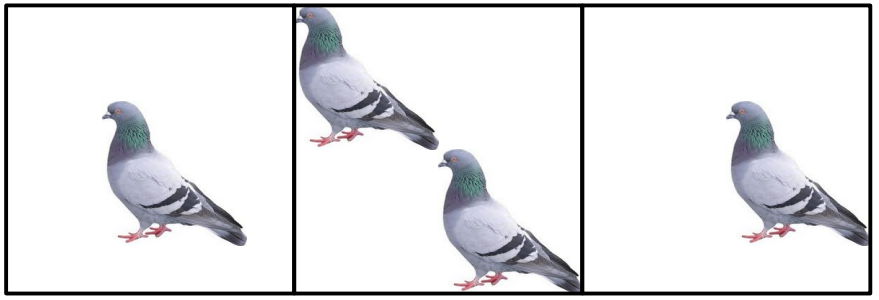

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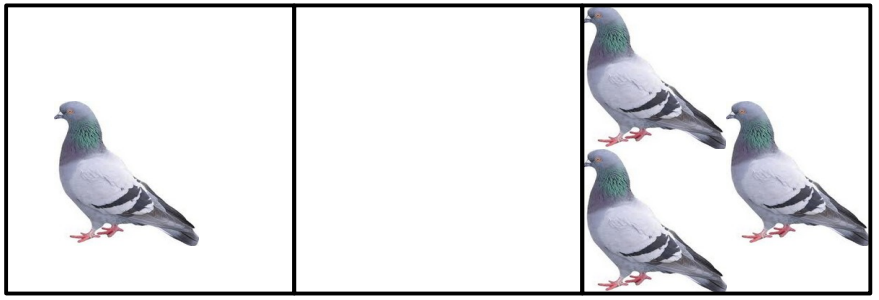

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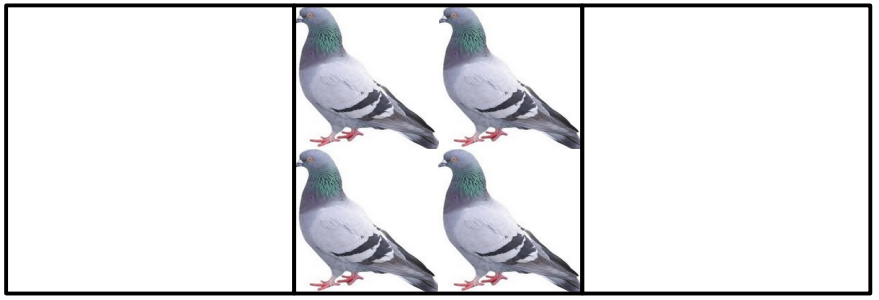
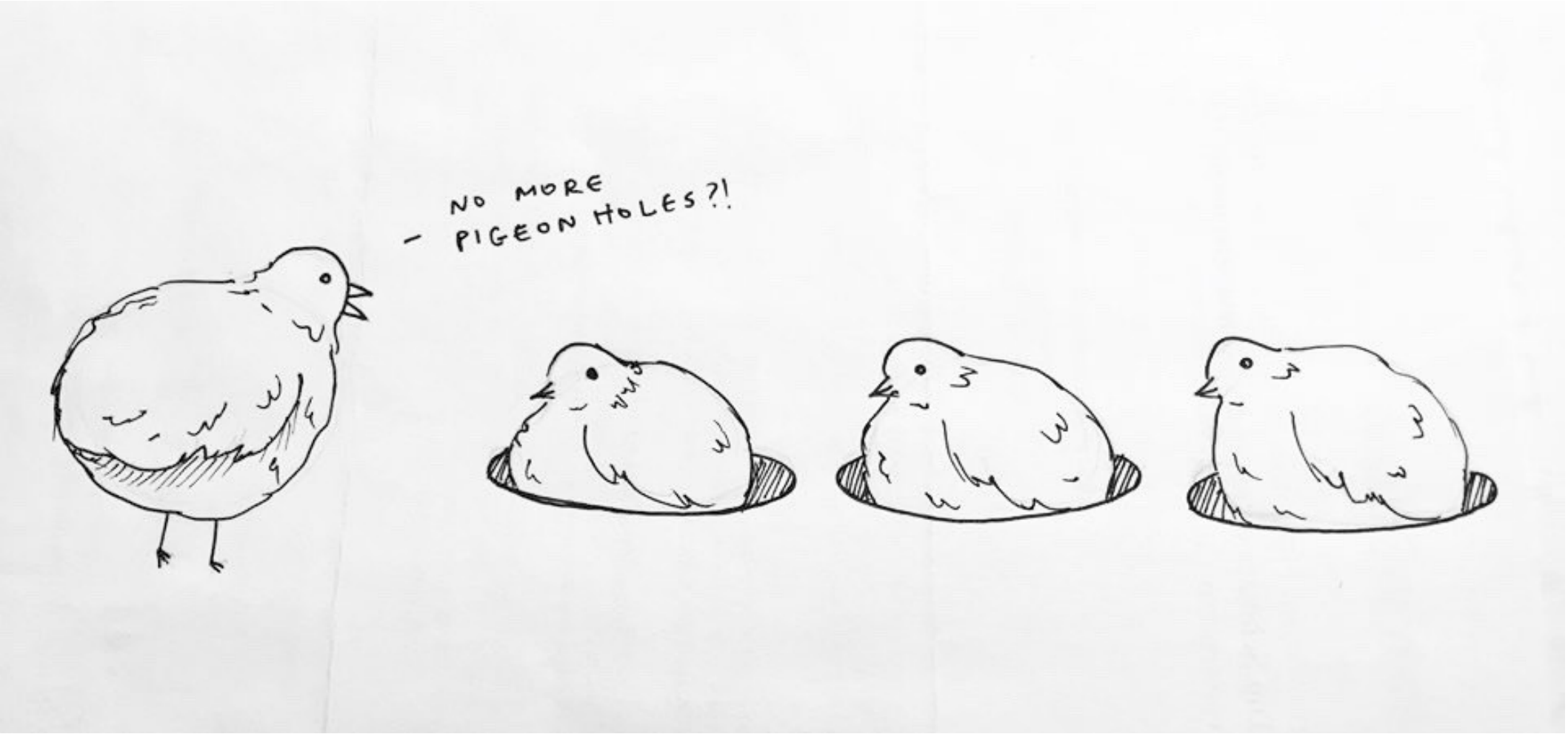




$m = 4, n = 3$

Thanks to Amy Liu for this awesome drawing!

Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes).
  - 367 people (pigeons).
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.

Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.

Let A and B be finite sets (sets whose cardinalities are natural numbers) and assume $|A| > |B|$. Which of the following statements are true for all functions $f : A \to B$?

(1) f is injective.
(2) f is not injective.
(3) f is surjective.
(4) f is not surjective.

Answer at

https://cs103.stanford.edu/pollev

Proving the Pigeonhole Principle

Theorem: If m objects are distributed into n bins and $m > n$, then there must be some bin that contains at least two objects.

Proof: Suppose for the sake of contradiction that, for some m and n where $m > n$, there is a way to distribute m objects into n bins such that each bin contains at most one object.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. There are m objects in total, so we know that

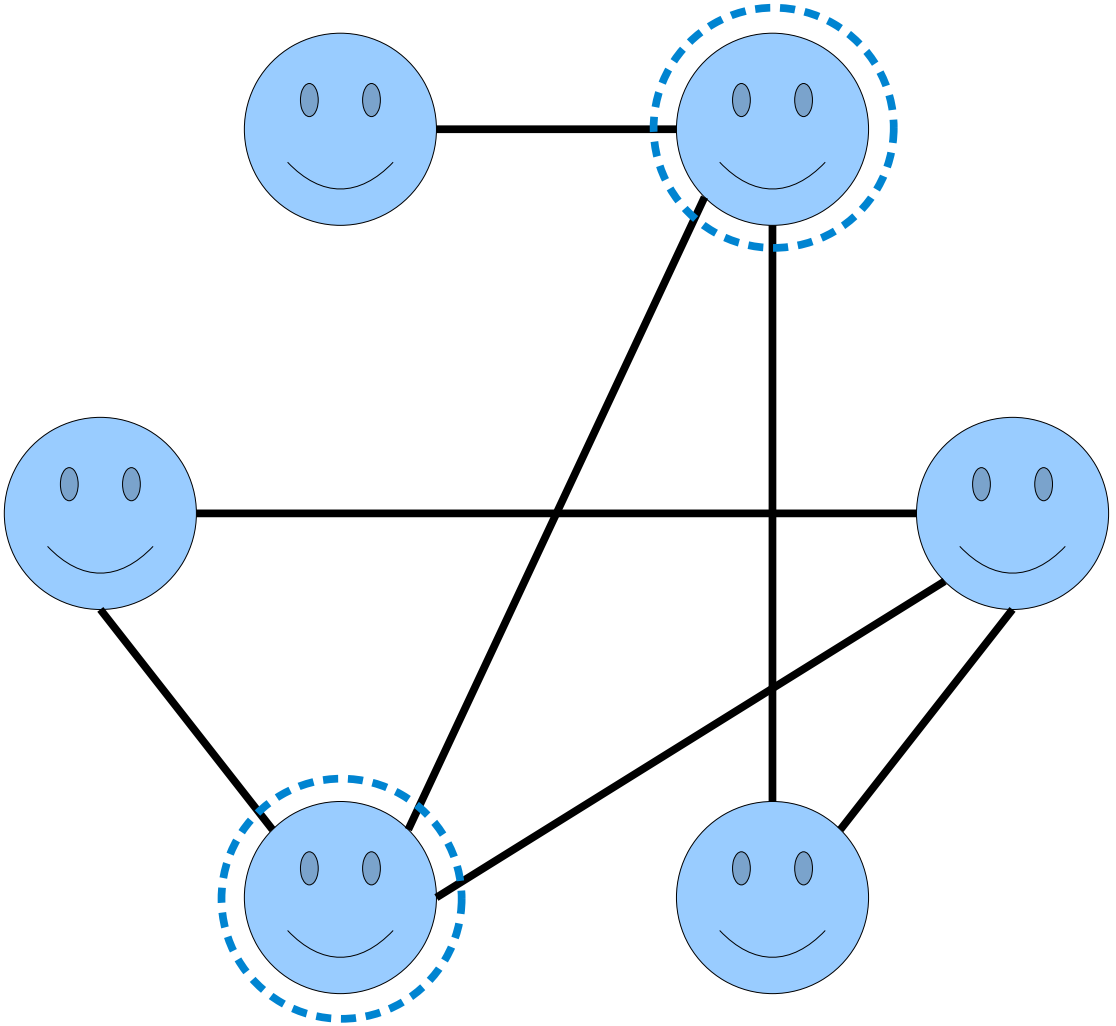
$$m = x_1 + x_2 + \ldots + x_n.$$

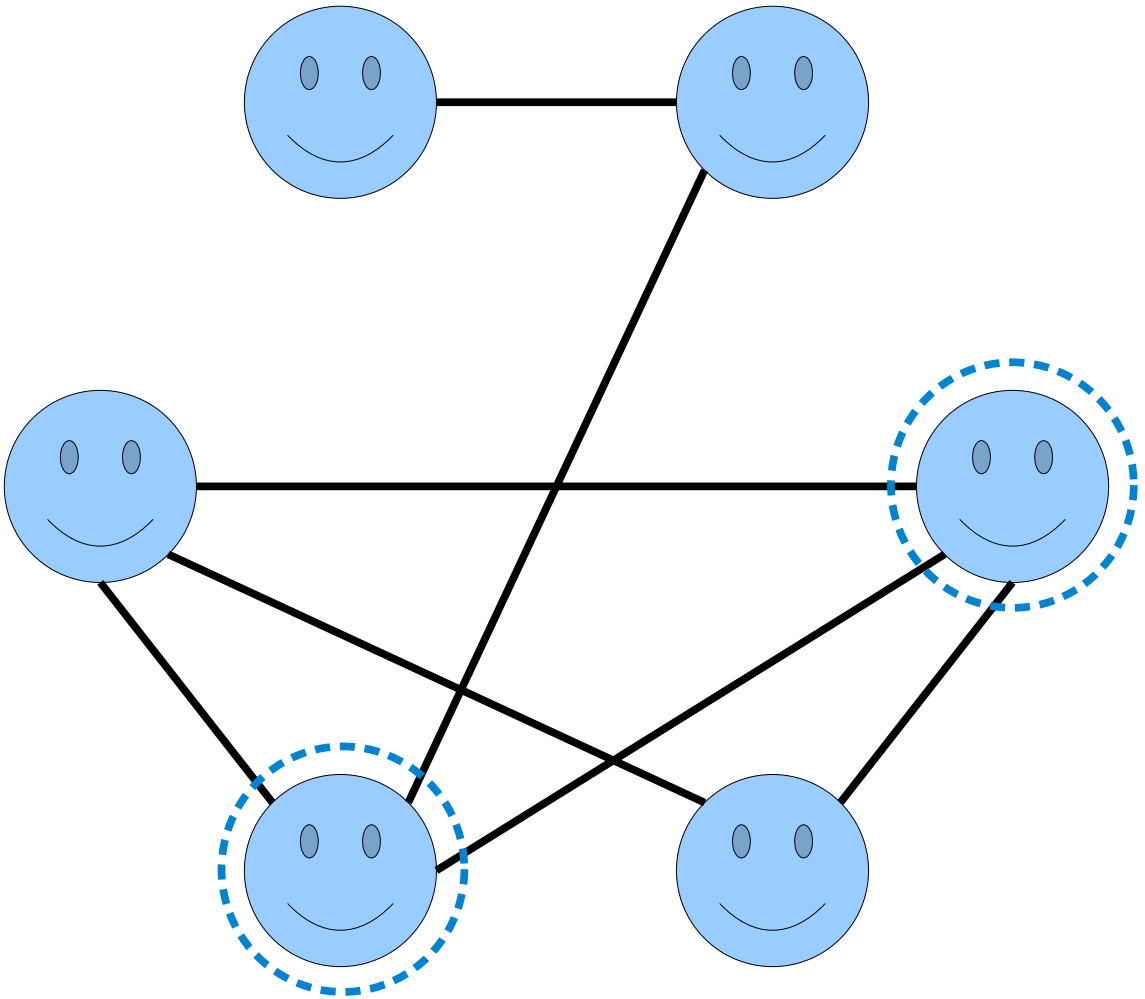
Since each bin has at most one object in it, we know $x_i \leq 1$ for each i. This means that

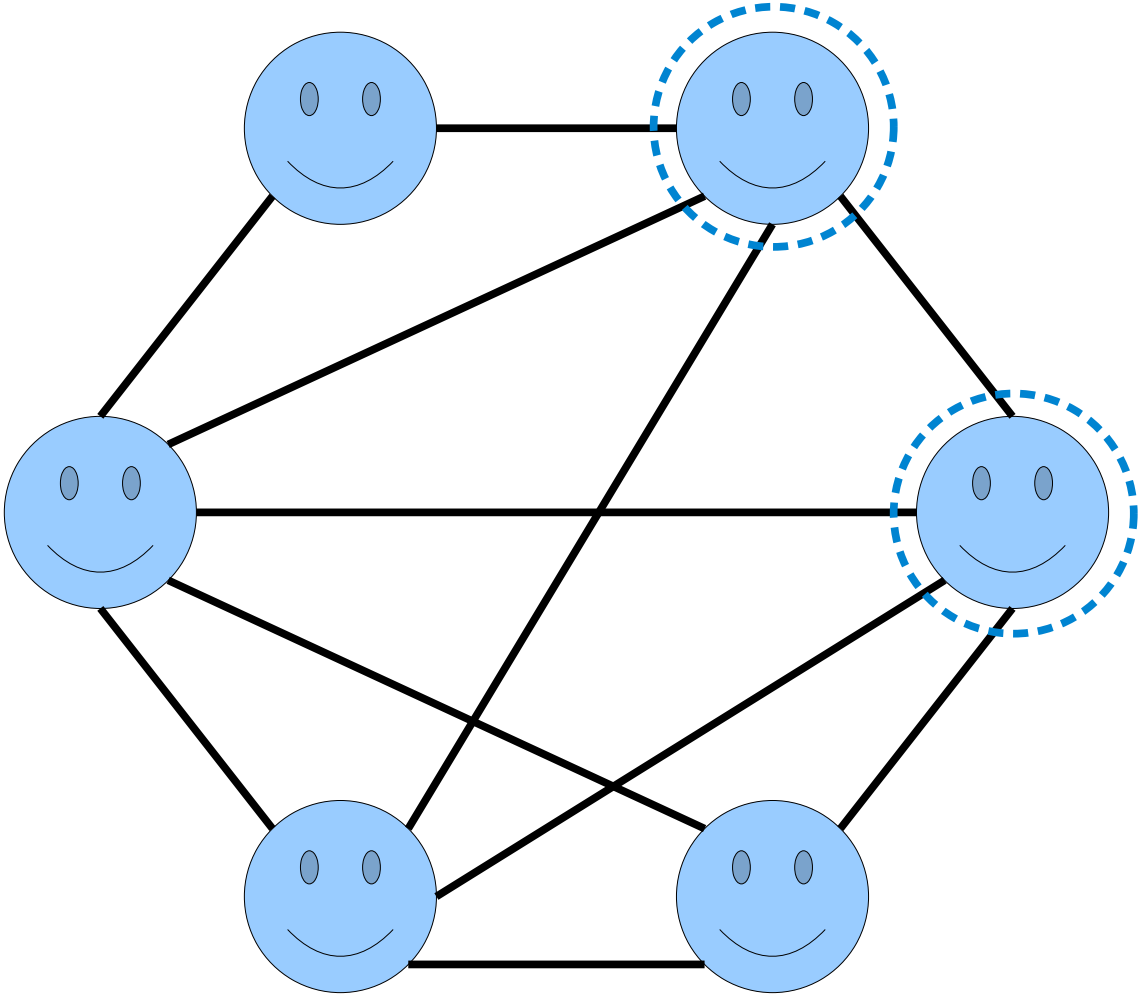
$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &\leq 1 + 1 + \ldots + 1 \quad \textit{(n times)} \\ &= n. \end{aligned}$$

This means that $m \leq n$, contradicting that $m > n$. We've reached a contradiction, so our assumption must have been wrong. Therefore, if m objects are distributed into n bins with $m > n$, some bin must contain at least two objects. ■

Pigeonhole Principle Party Tricks





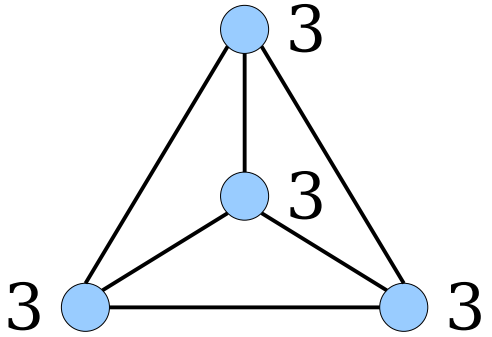


Hmm…. Is this a guarantee?

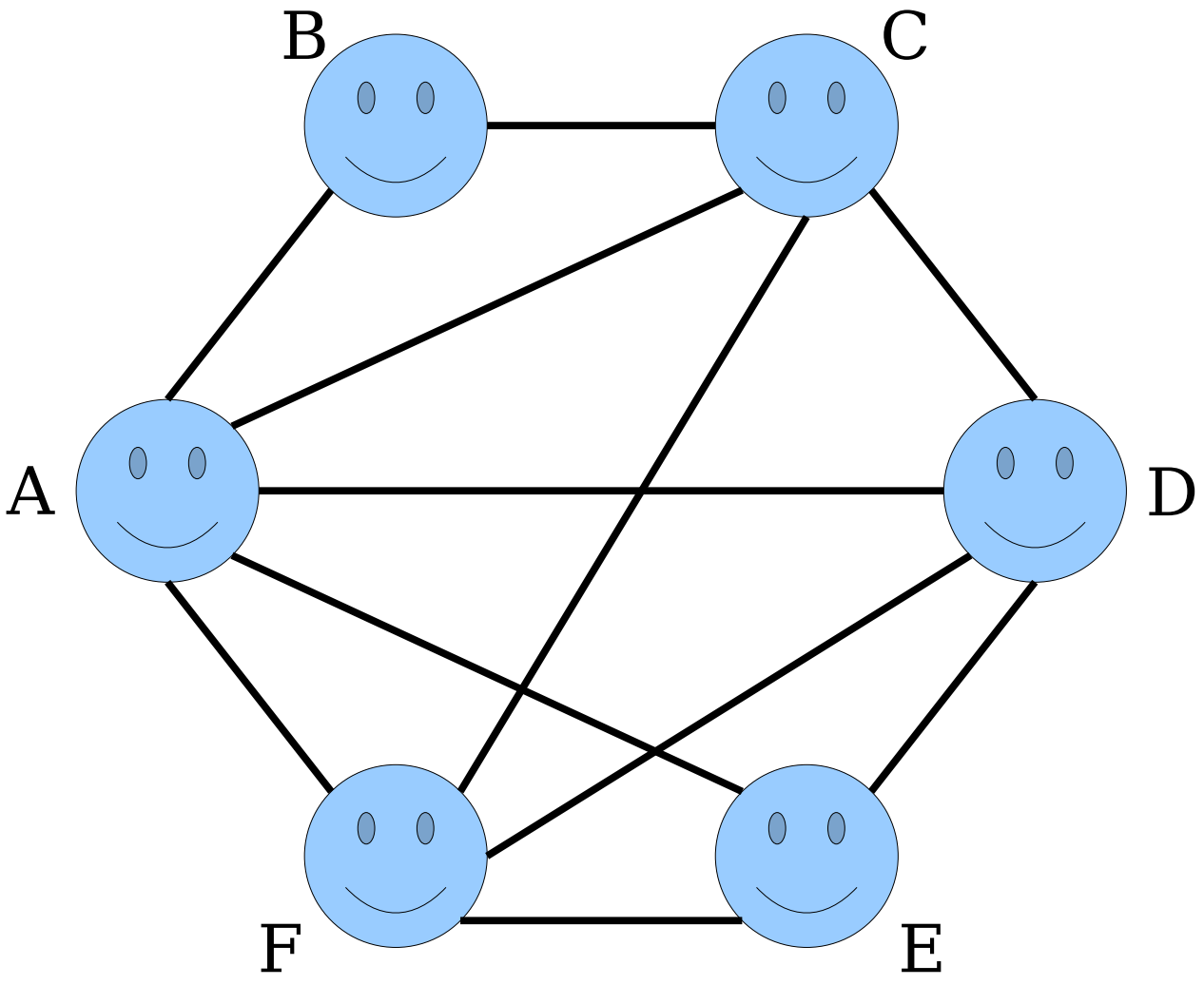
Let’s explore the idea mathematically!

Degrees

- The ***degree*** of a node v in a graph is the number of nodes that v is adjacent to.



- ***Theorem:*** Every graph with at least two nodes has at least two nodes with the same degree.
  - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.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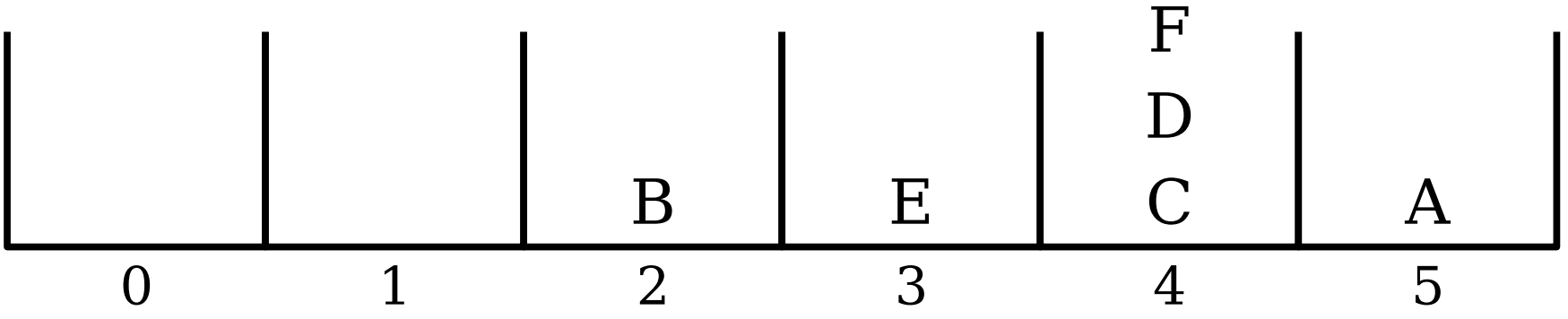
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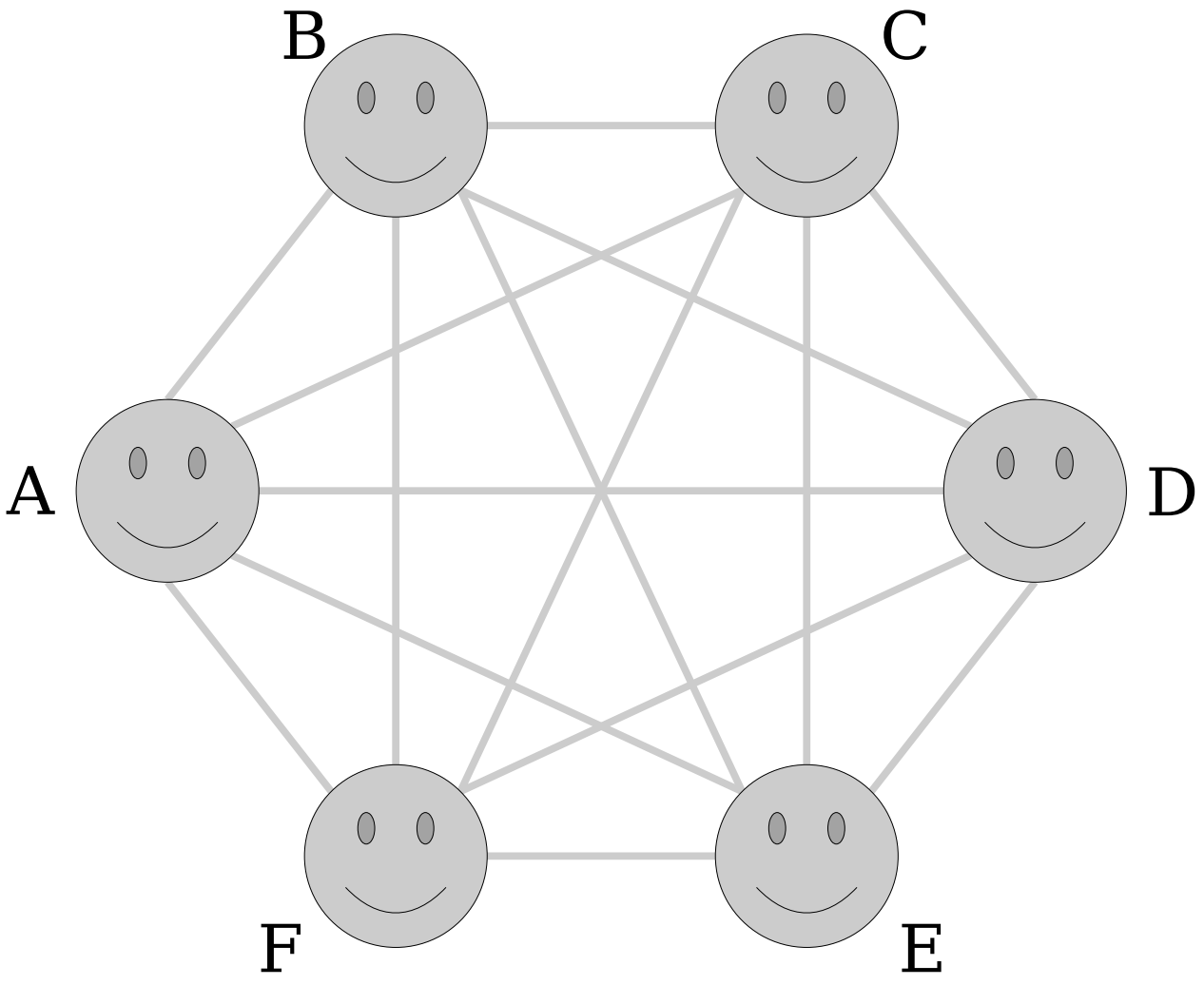
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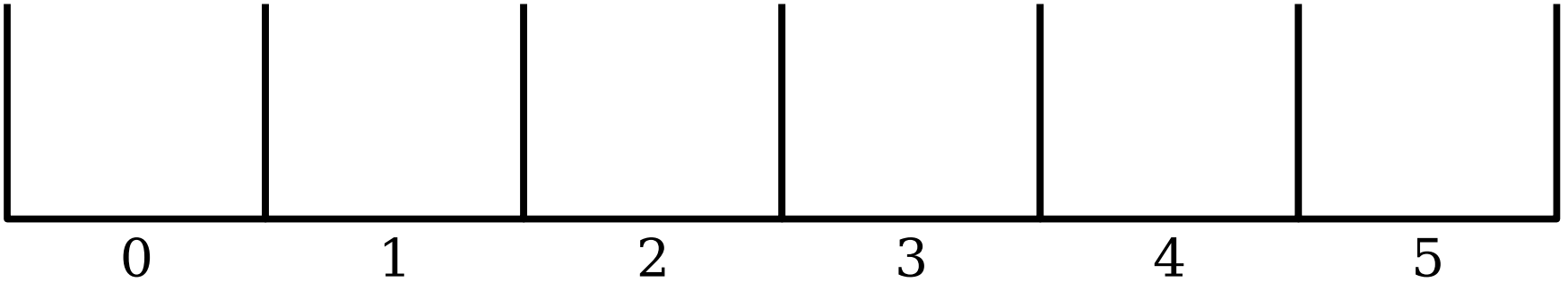
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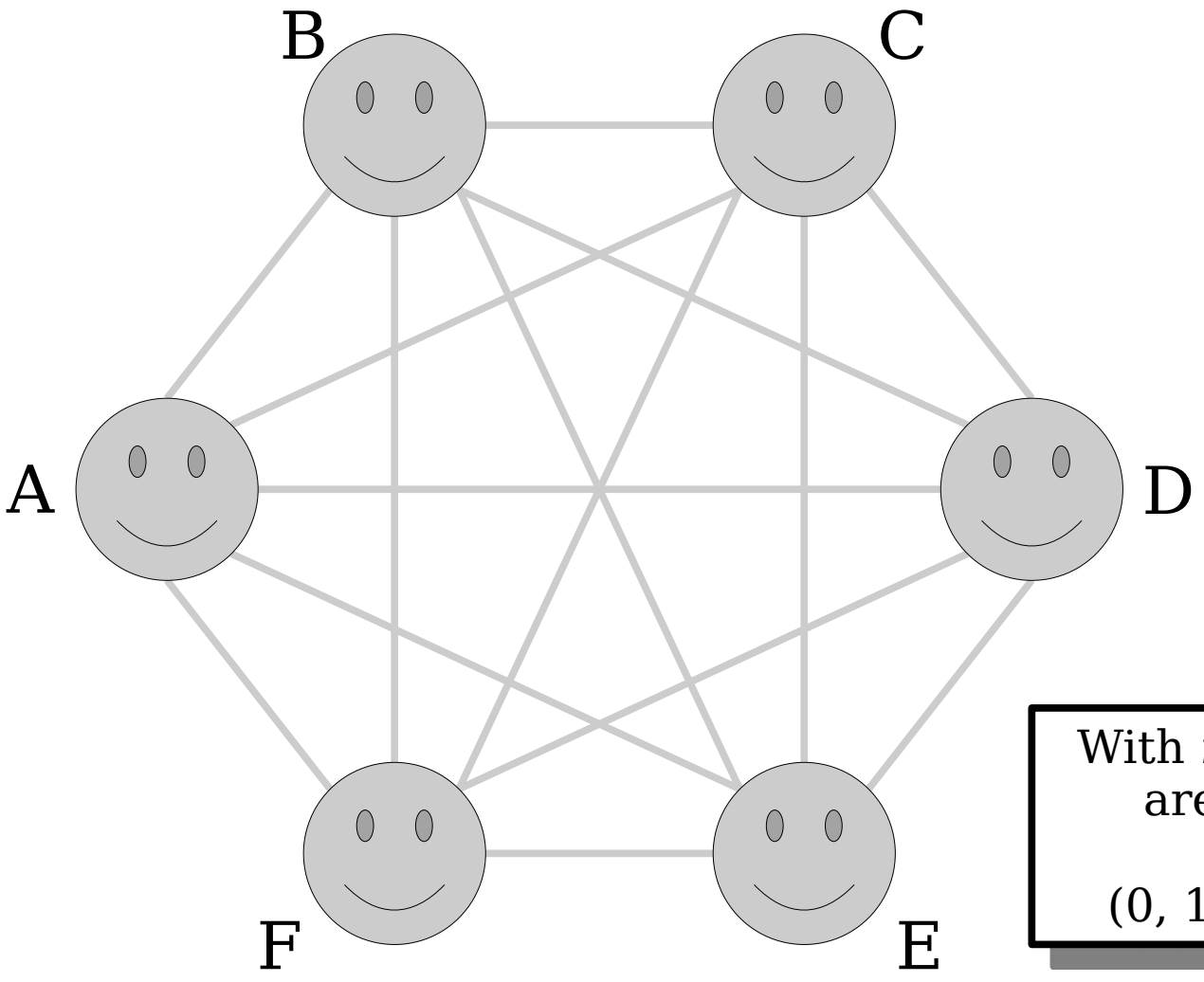
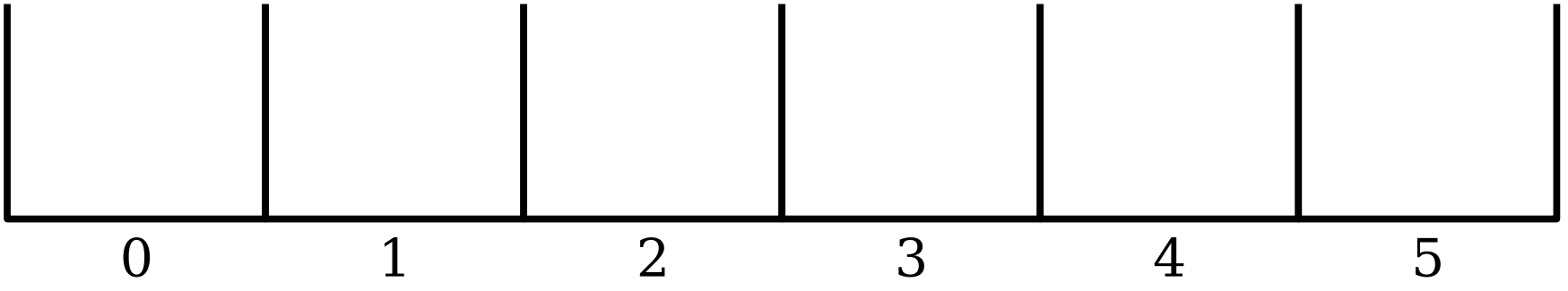
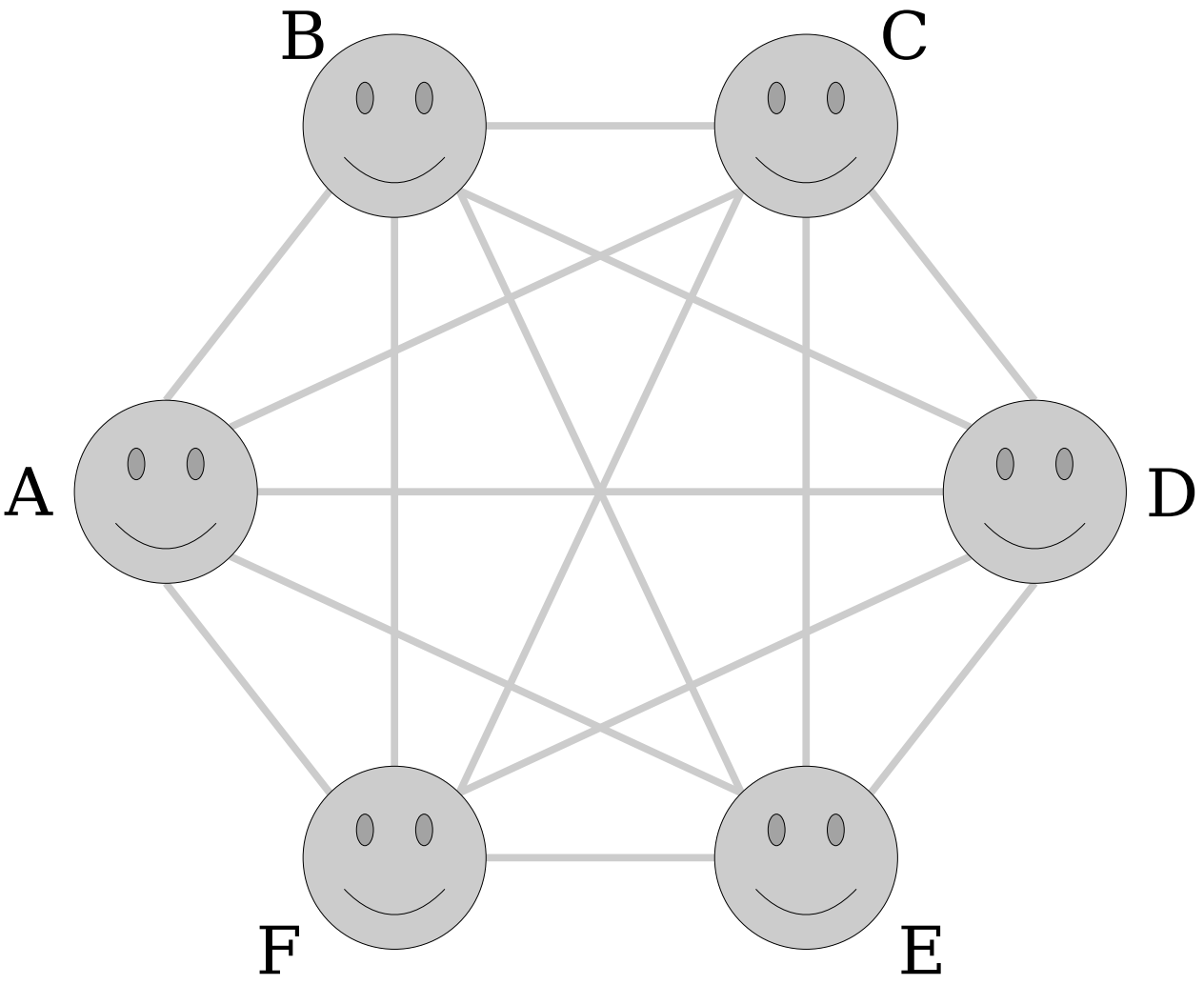


With n nodes, there are n possible degrees
(0, 1, 2, …, n – 1)

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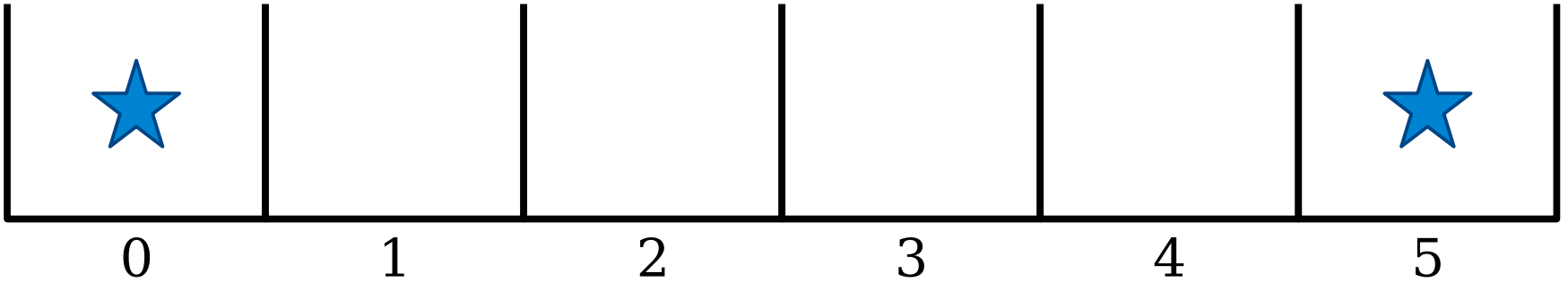
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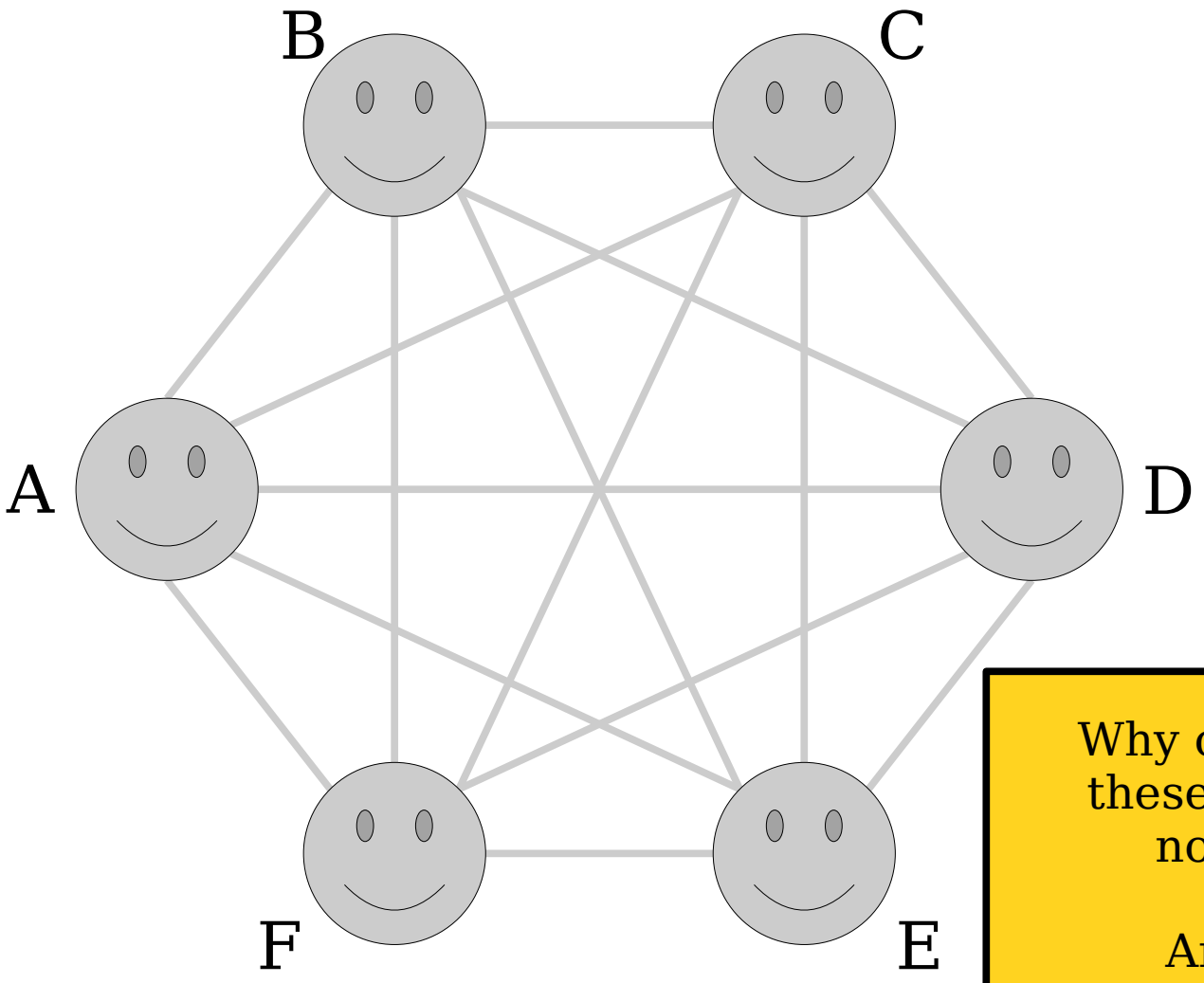
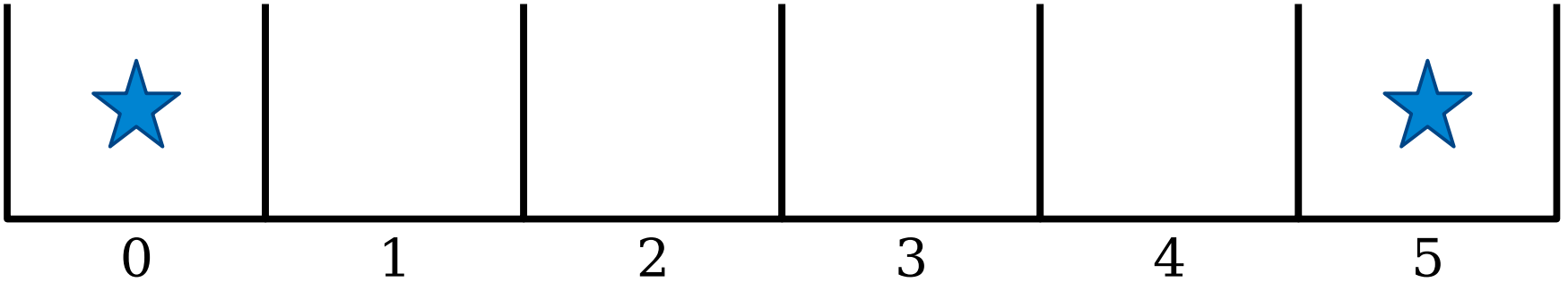
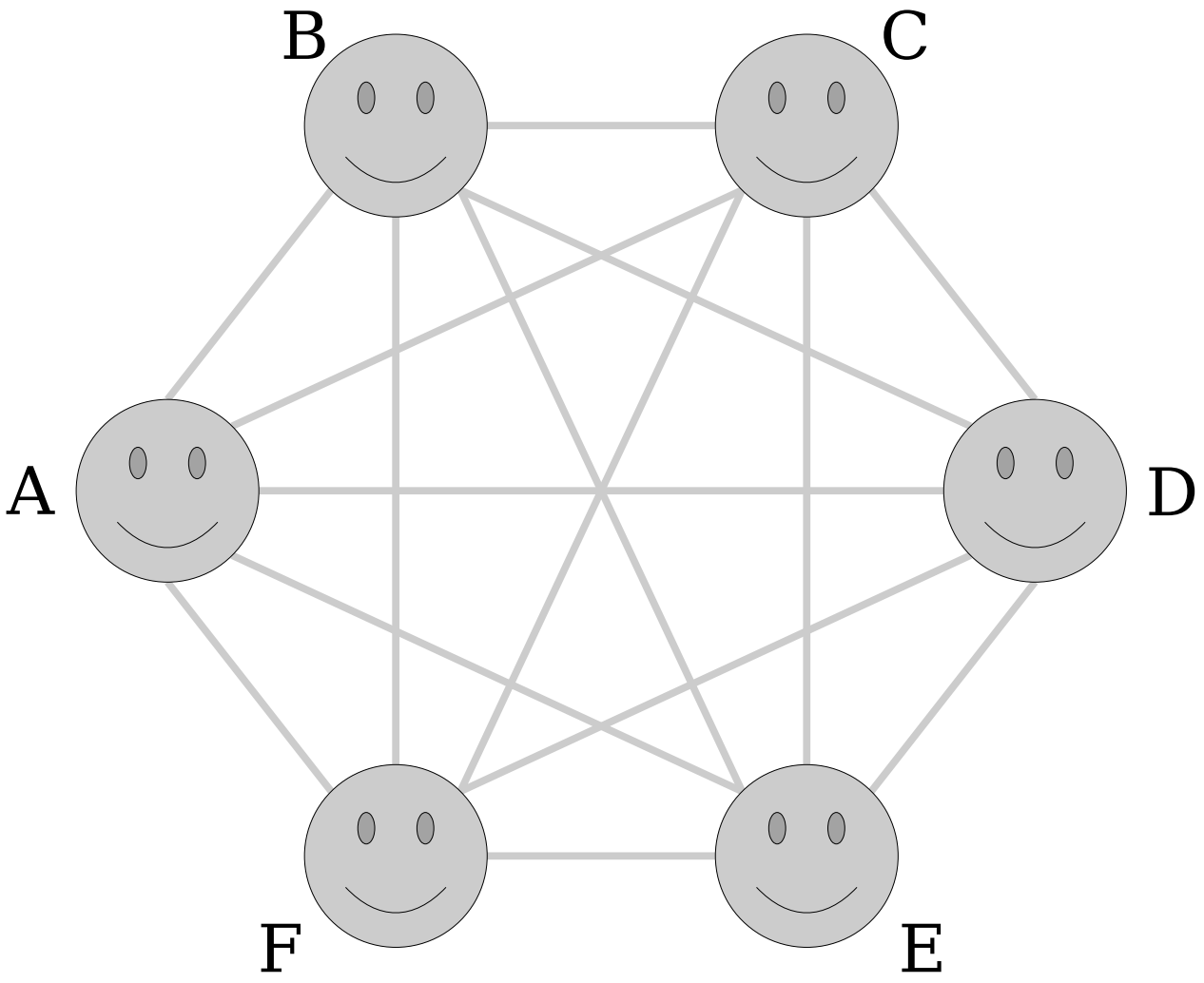


Why can't both of these buckets be nonempty?

Answer at
cs103.stanford.edu/pollev

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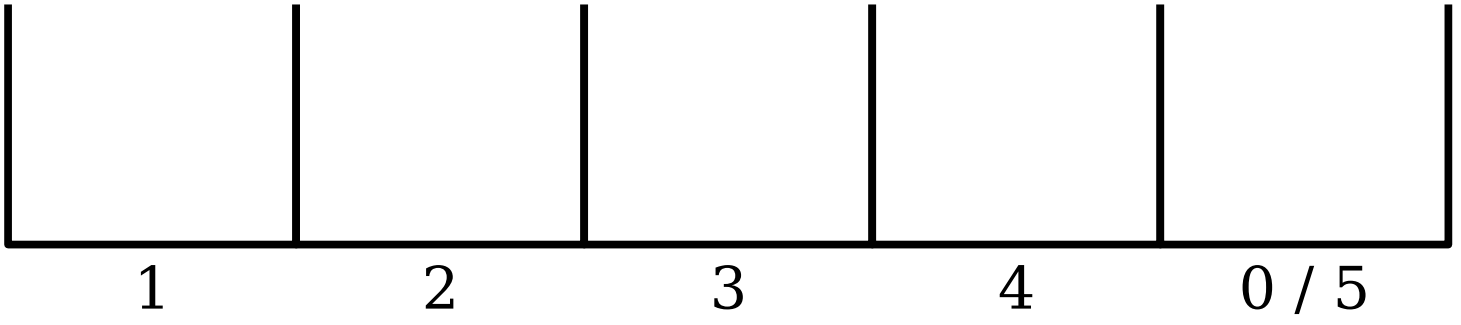
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Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G, namely 0, 1, 2, …, n – 1, so this means that G must have exactly one node of each degree. However, this means that G has a node u of degree 0 and a node v of degree n – 1. We see that $u \neq v$ because v has degree n – 1 ≥ 1 > 0. Since u has degree 0, we know $\{u, v\} \notin E$. But since v has degree n – 1 and there are only n – 1 nodes in V aside from v, we see that $\{u, v\} \in E$.

We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

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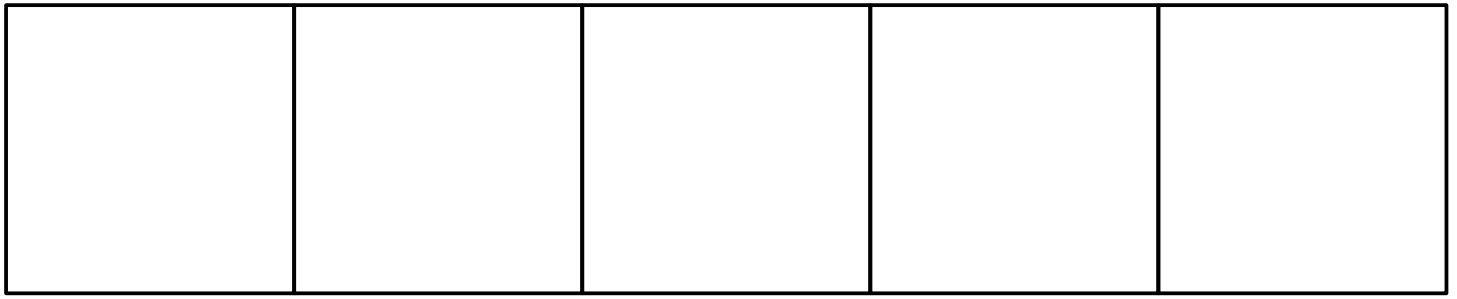

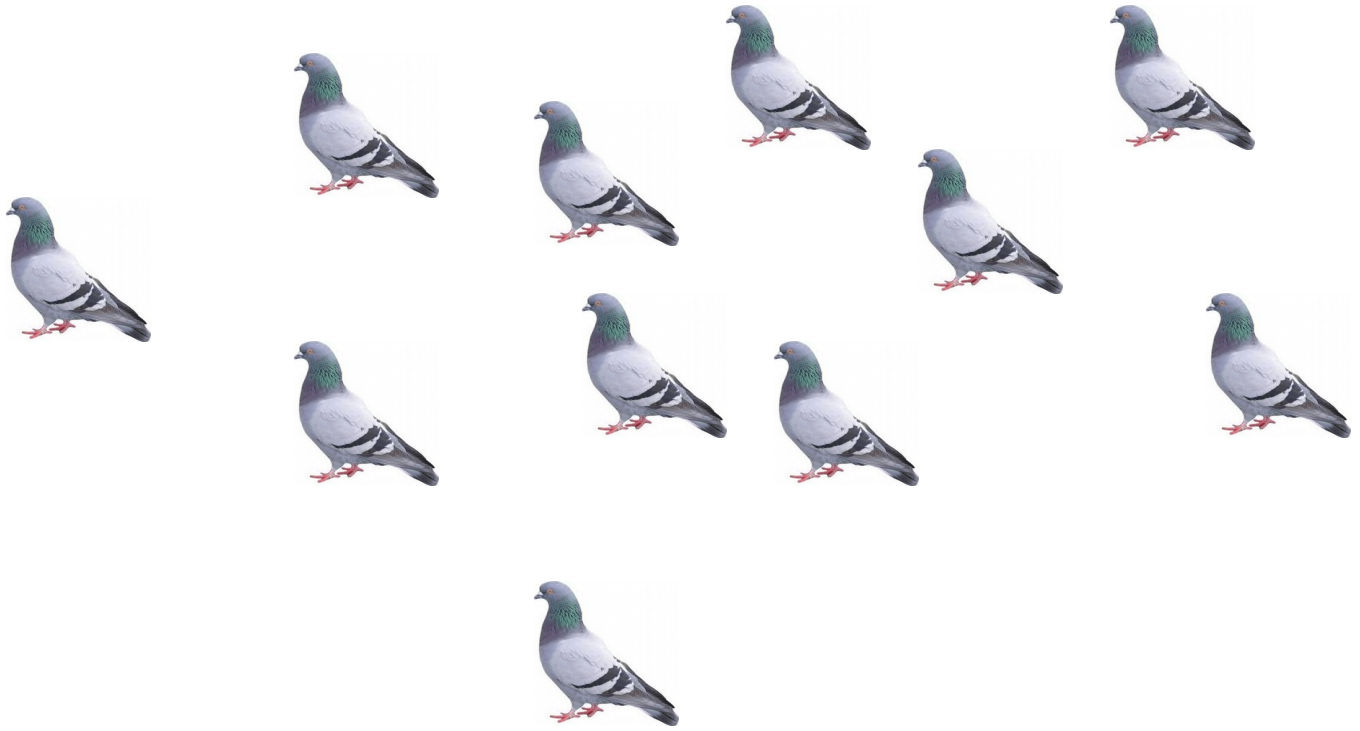

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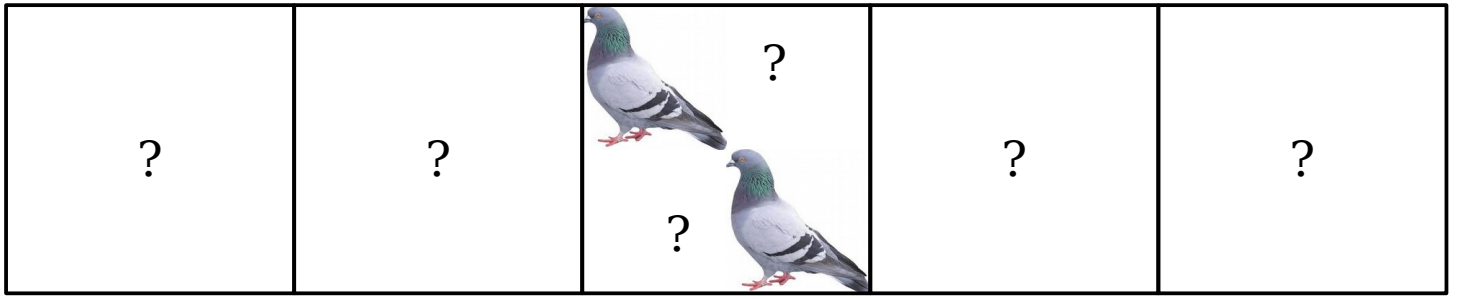

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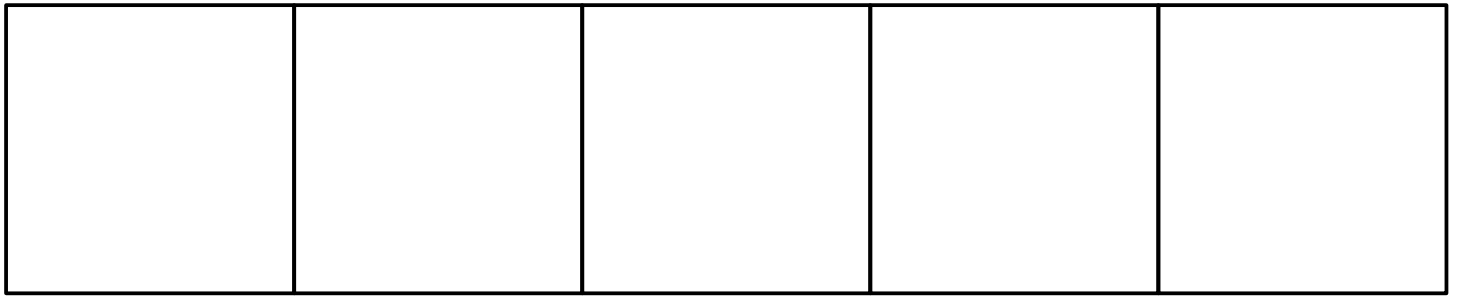

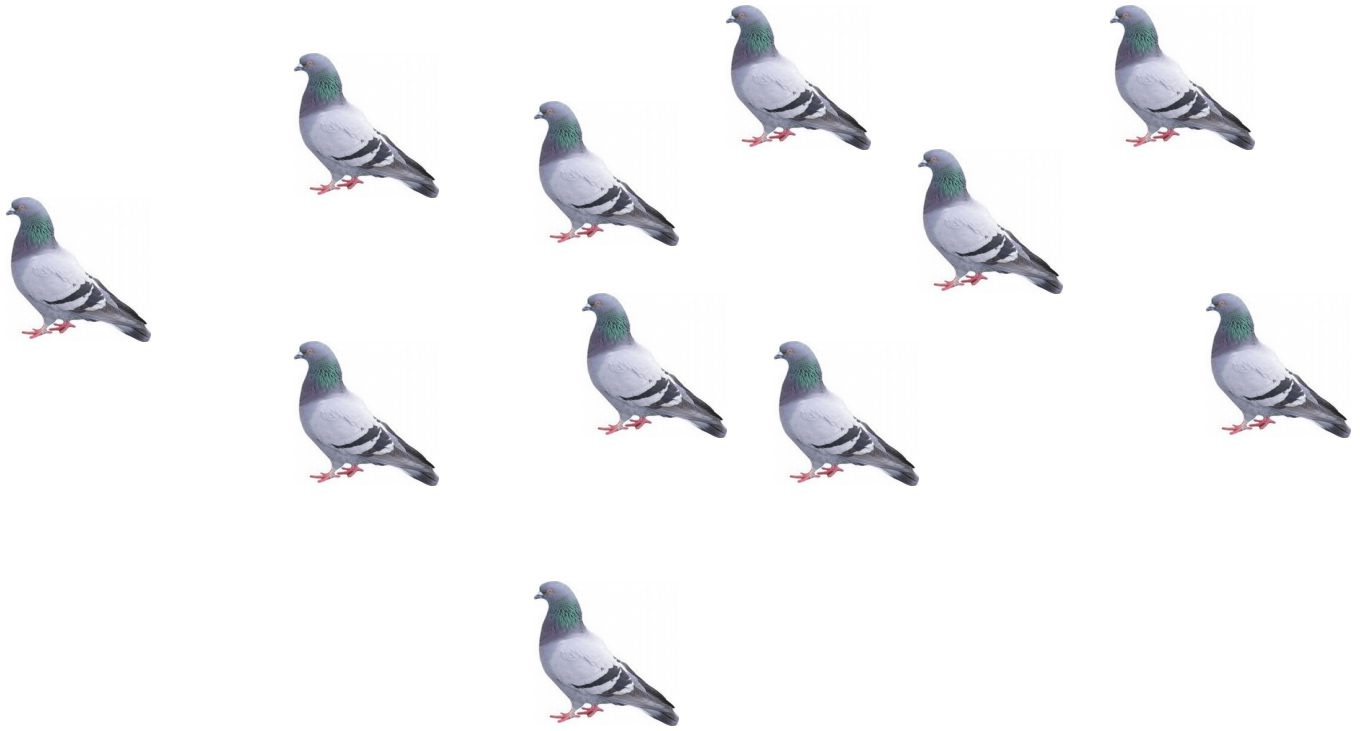

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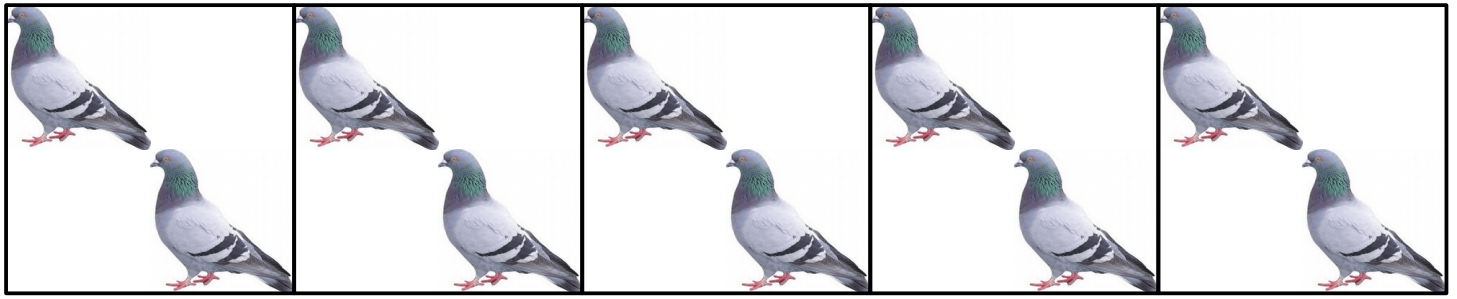

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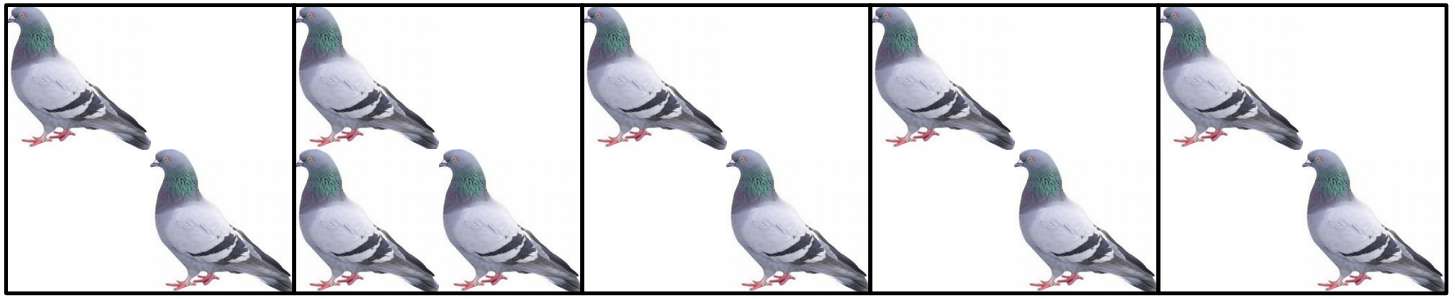

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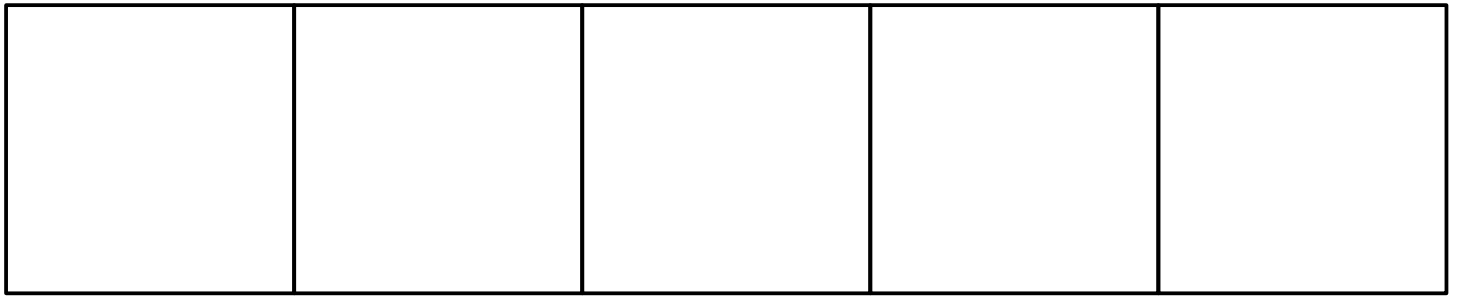

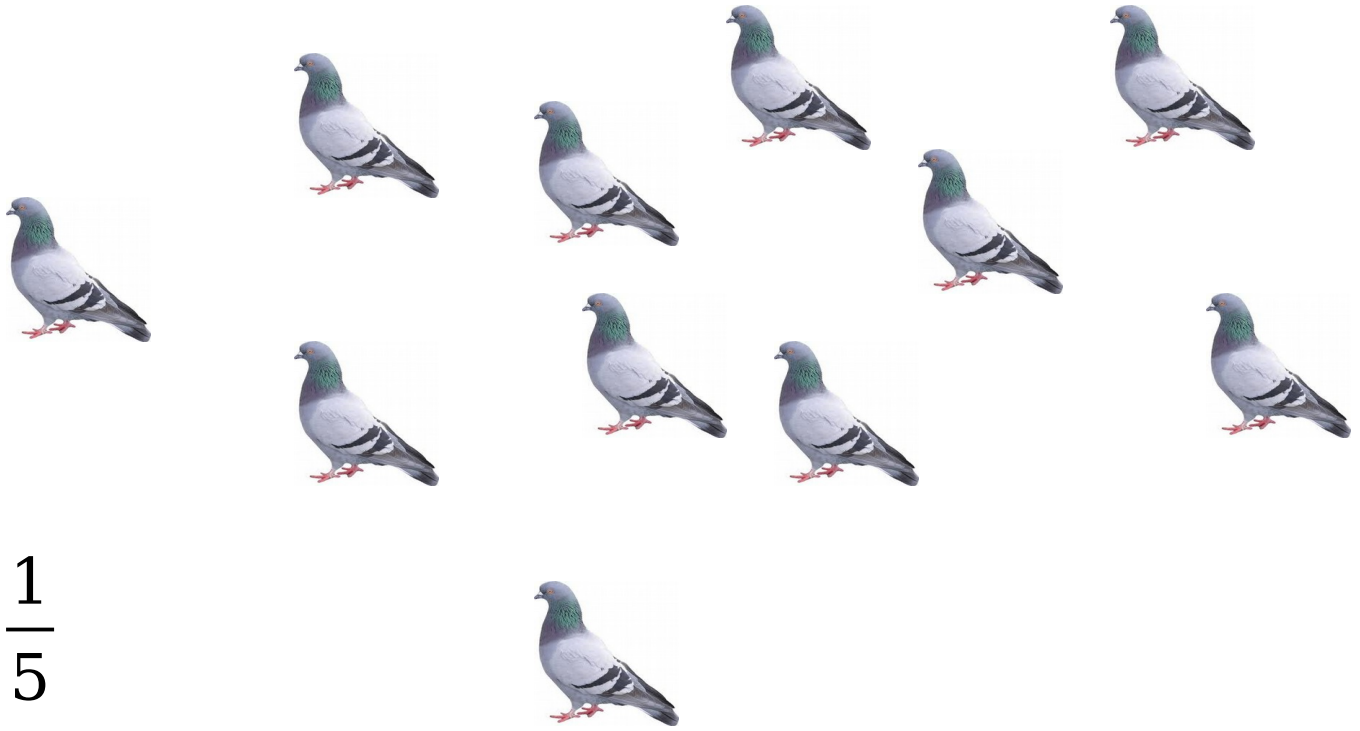

$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The ***generalized pigeonhole principle*** says that if you distribute m objects into n bins, then
  - some bin will have at least $\lceil m/n \rceil$ objects in it, and
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$\lceil m/n \rceil$ means "m/n, rounded up."
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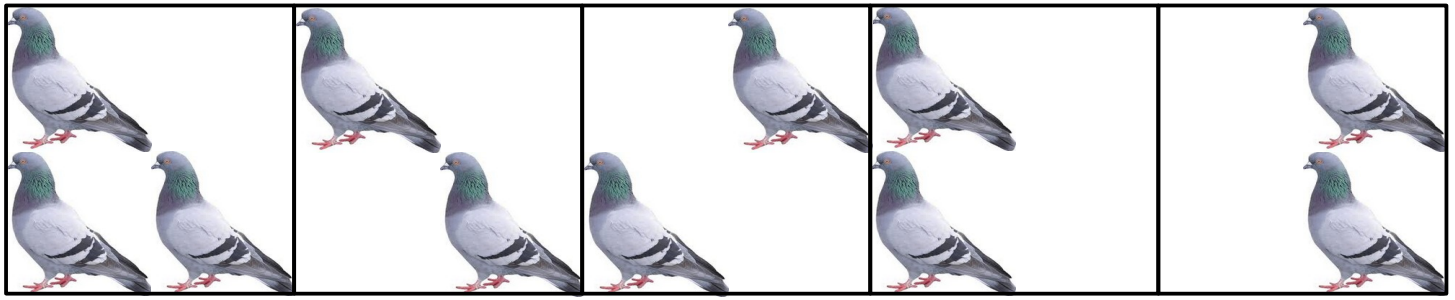

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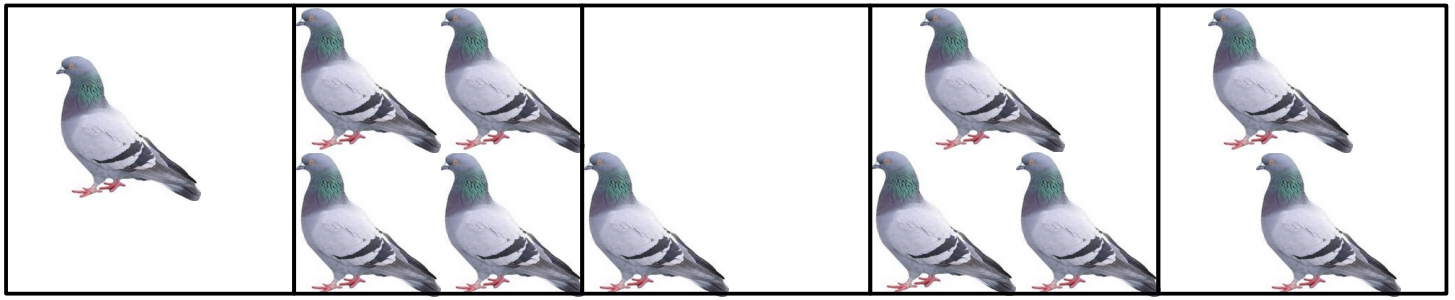

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$\lceil m/n \rceil$ means "m/n, rounded up."
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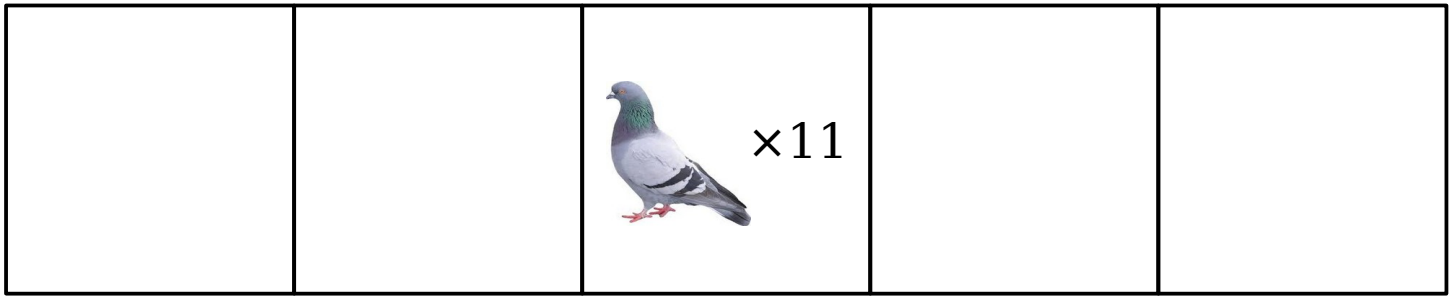
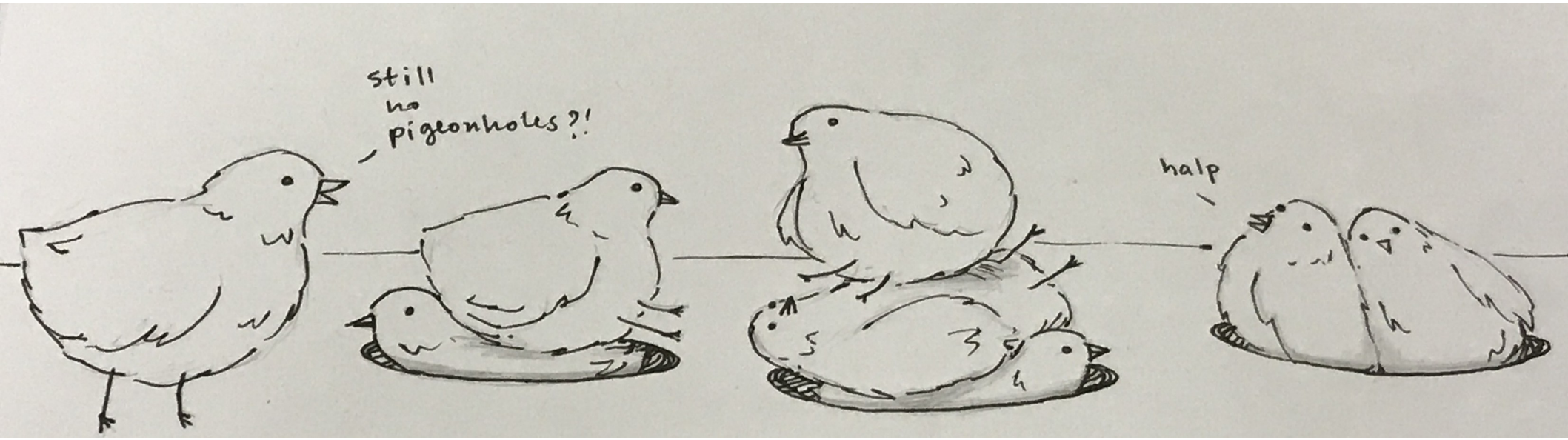


$m = 11$
$n = 5$

$\lceil m / n \rceil = 3$
$\lfloor m / n \rfloor = 2$



$m = 8, n = 3$

Thanks to Amy Liu for this awesome drawing!

Theorem: If m objects are distributed into $n > 0$ bins, then some bin will contain at least $\lceil m/n \rceil$ objects.

Proof: We will prove that if m objects are distributed into n bins, then some bin contains at least m/n objects. Since the number of objects in each bin is an integer, this will prove that some bin must contain at least $\lceil m/n \rceil$ objects.

To do this, we proceed by contradiction. Suppose that, for some m and n, there is a way to distribute m objects into n bins such that each bin contains fewer than m/n objects.

Number the bins 1, 2, 3, …, n and let x_i denote the number of objects in bin i. Since there are m objects in total, we know that

$$m = x_1 + x_2 + \ldots + x_n.$$

Since each bin contains fewer than m/n objects, we see that $x_i < m/n$ for each i. Therefore, we have that

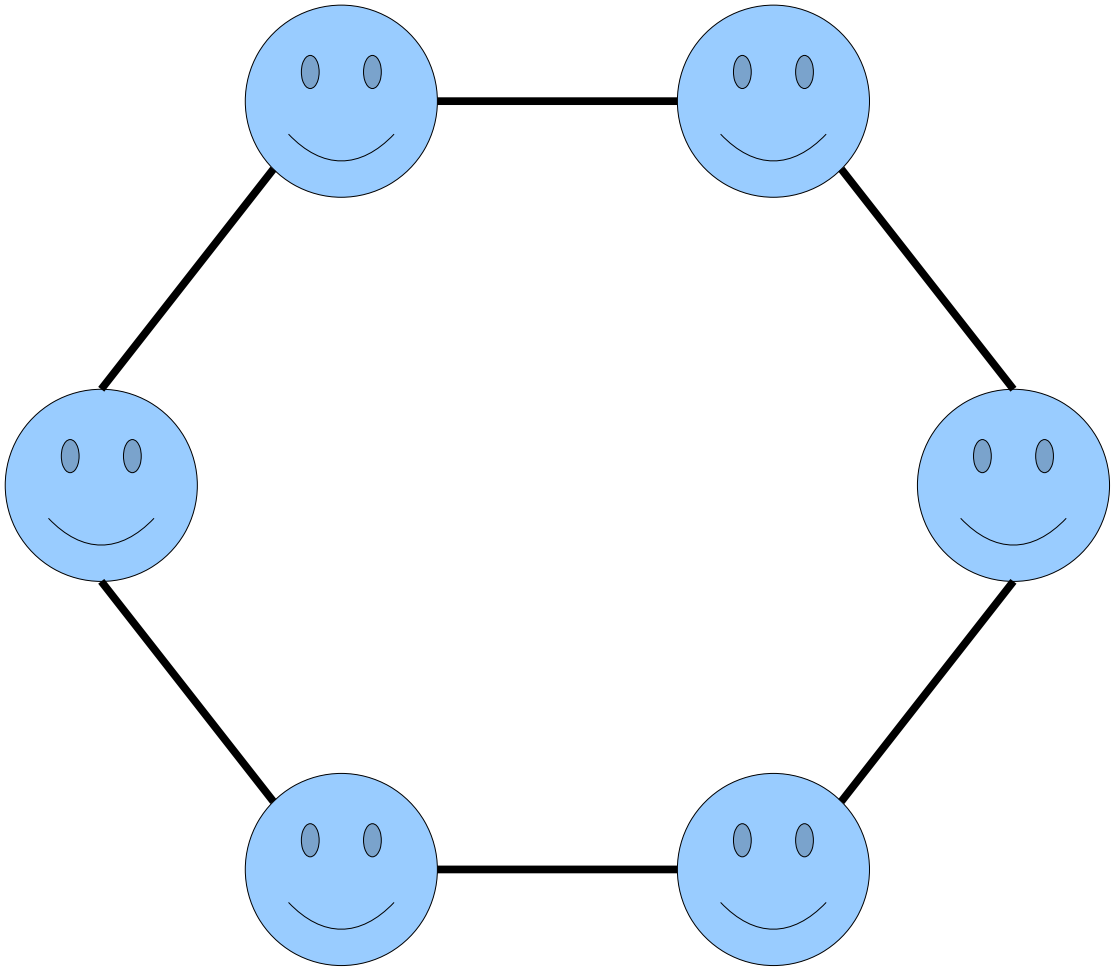
$$\begin{aligned} m &= x_1 + x_2 + \ldots + x_n \\ &< m/n + m/n + \ldots + m/n \quad \textit{(n times)} \\ &= m. \end{aligned}$$

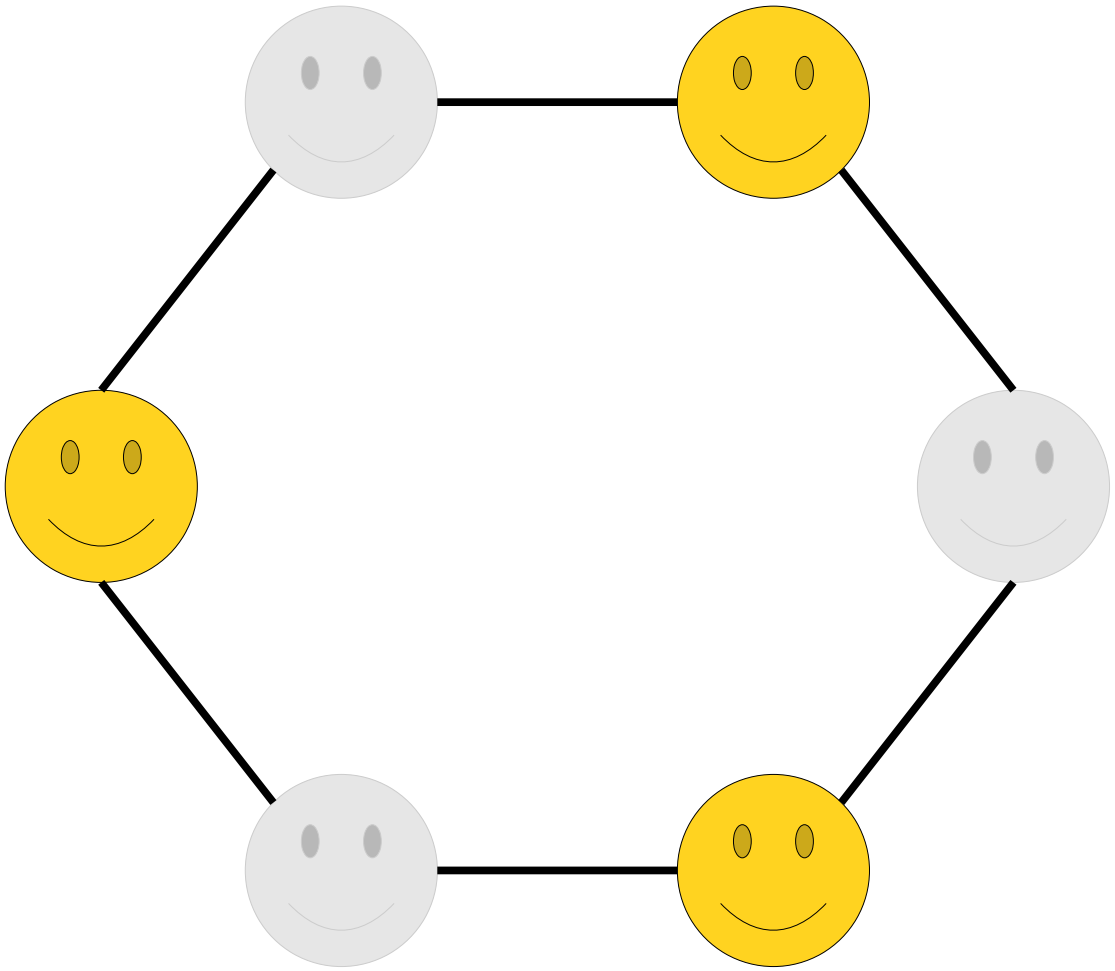
But this means that $m < m$, which is impossible. We have reached a contradiction, so our initial assumption must have been wrong. Therefore, if m objects are distributed into n bins, some bin must contain at least $\lceil m/n \rceil$ objects. ■

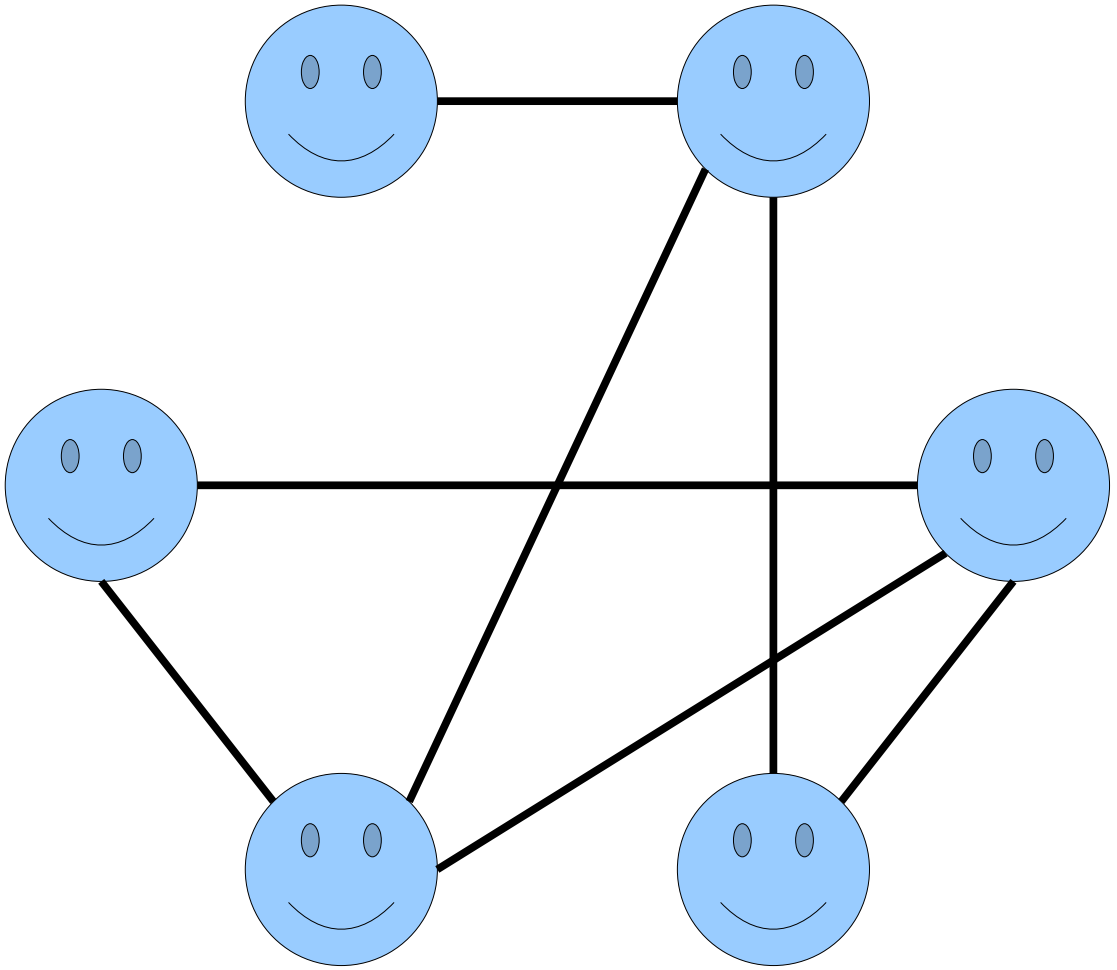
An Application: Friends and Strangers

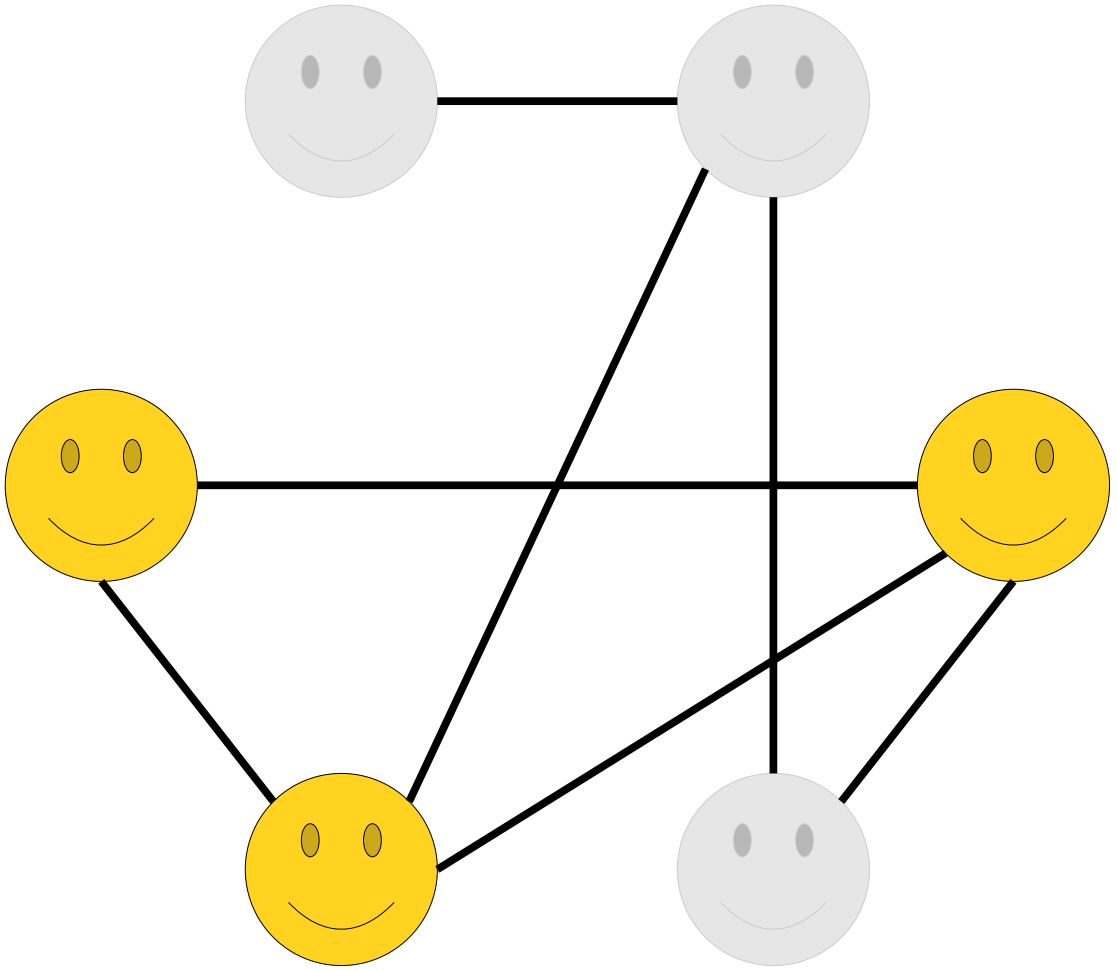
Friends and Strangers

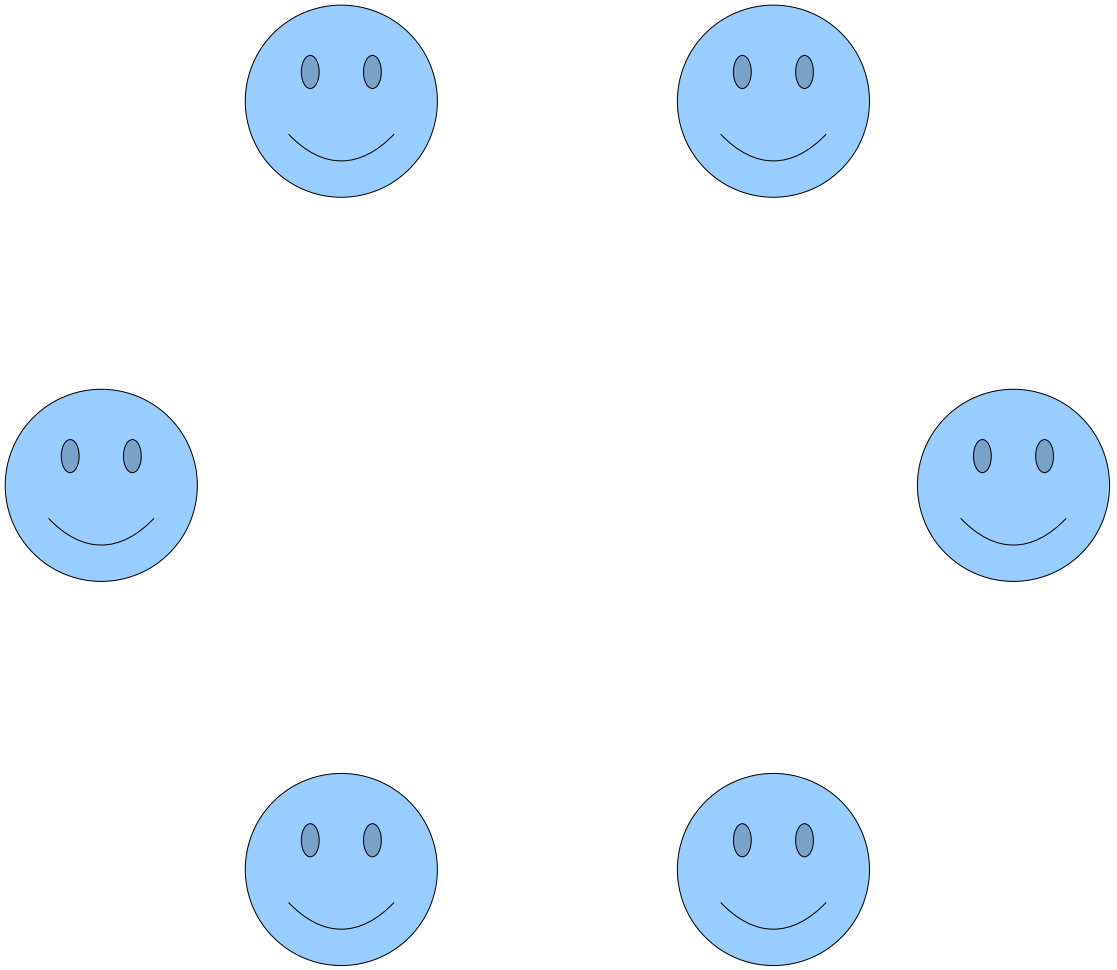
- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem ("Theorem on Friends and Strangers"):*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

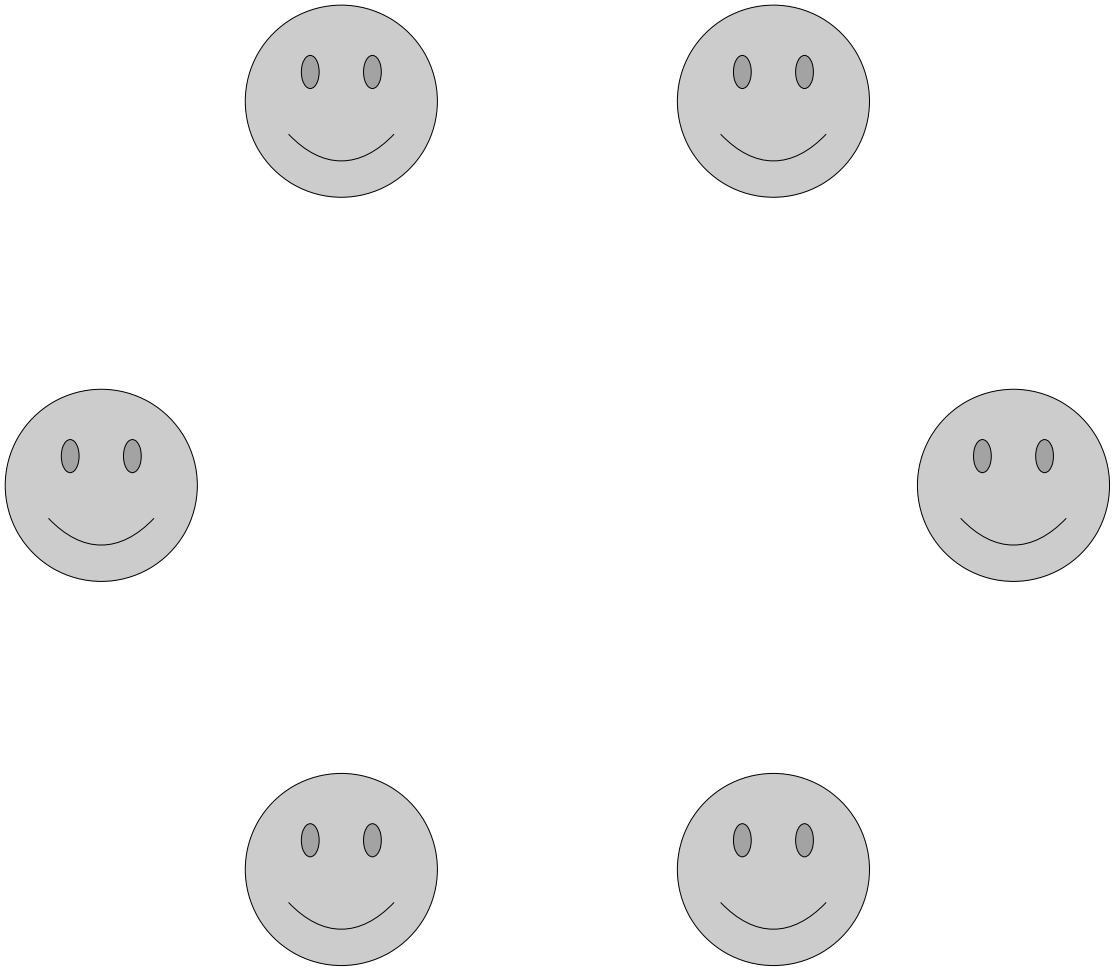


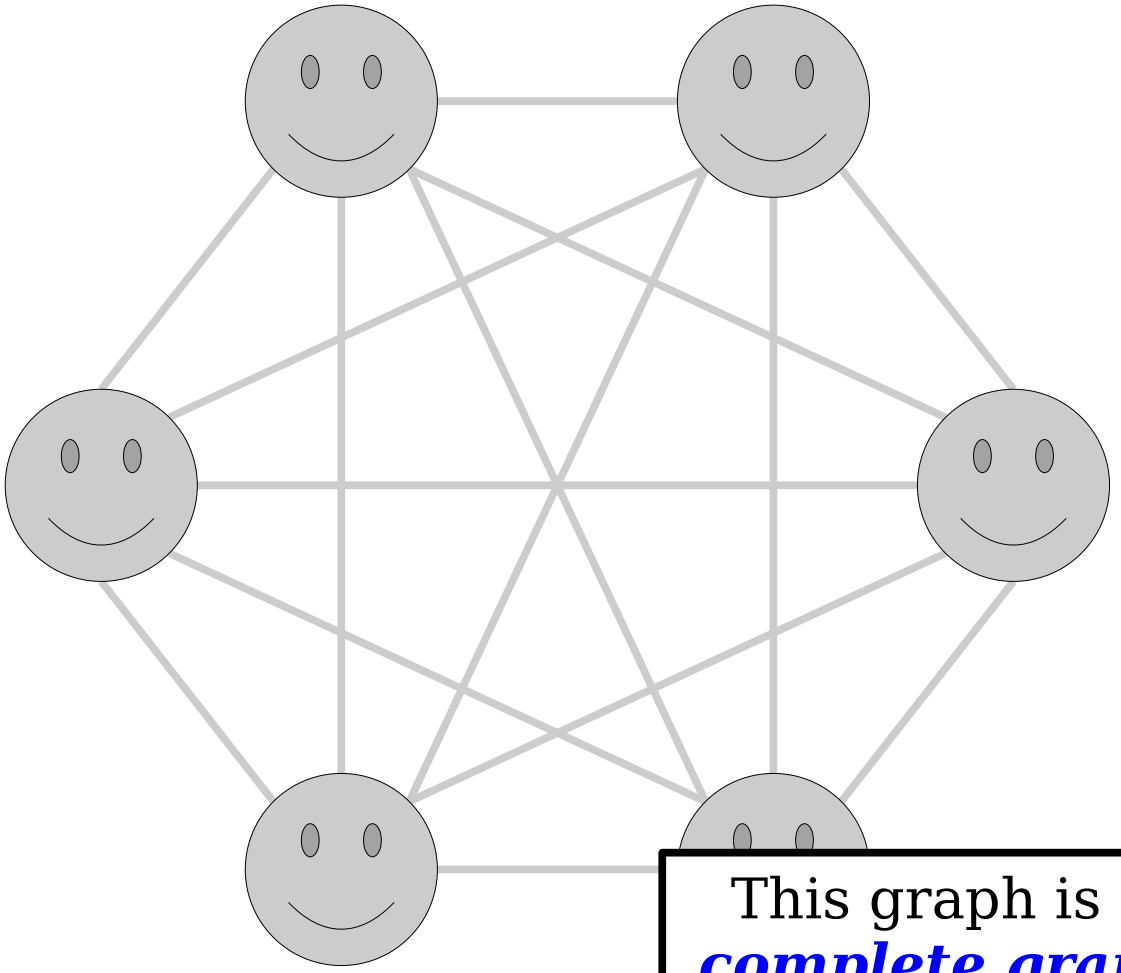




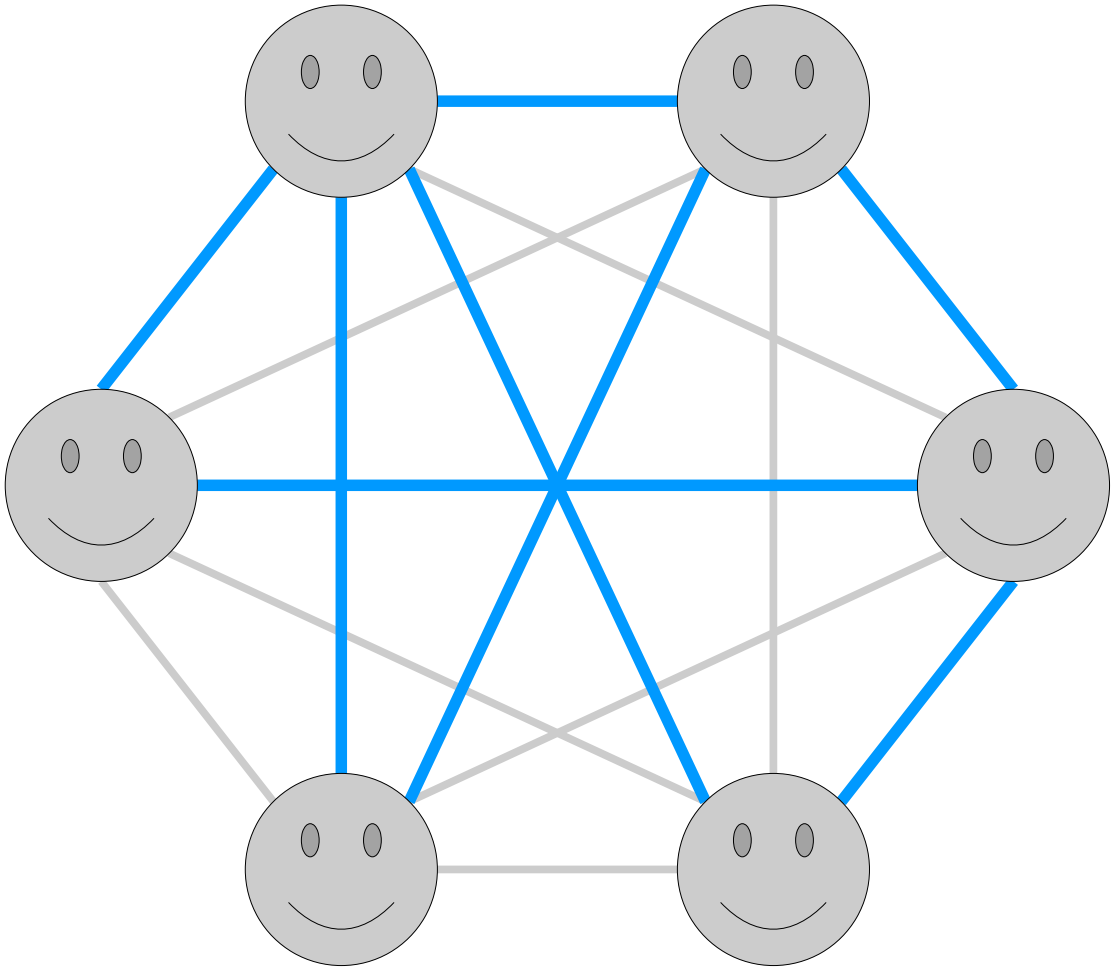


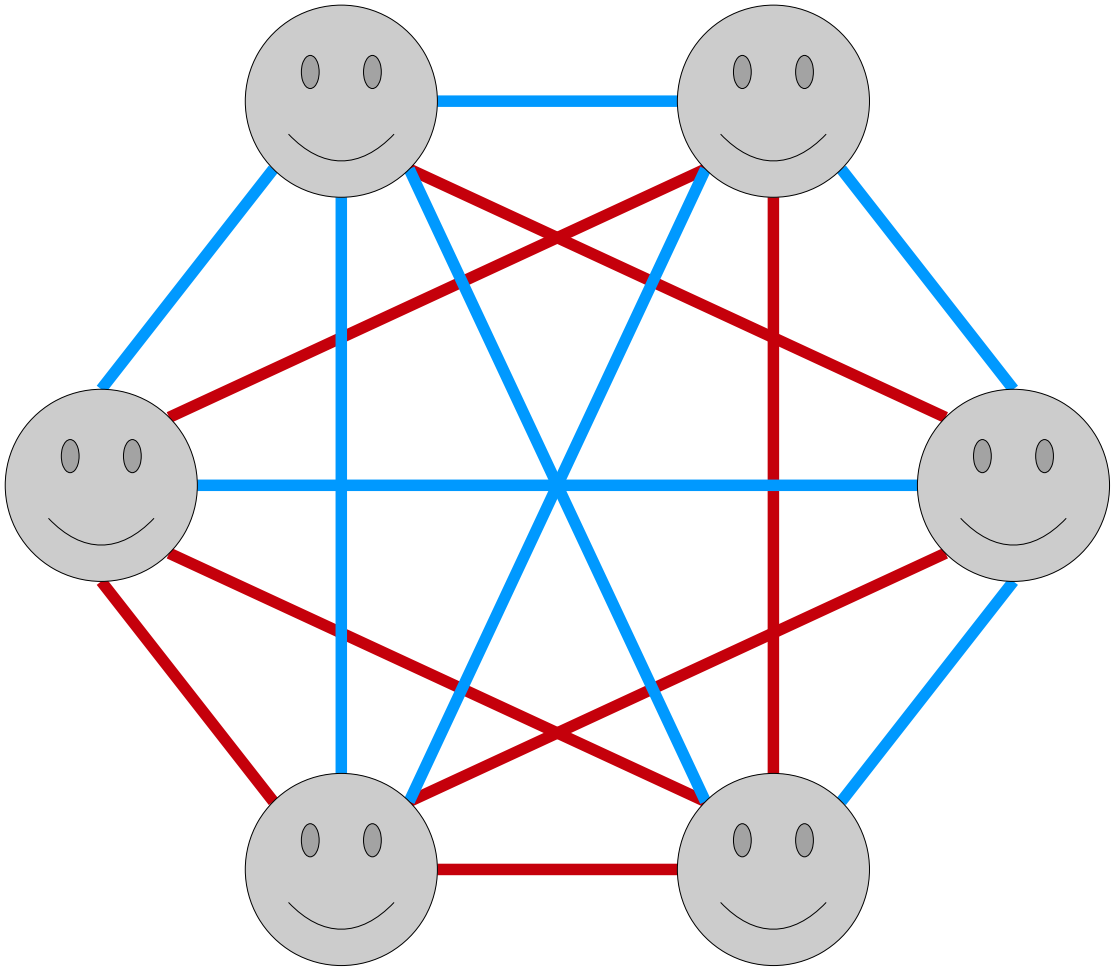


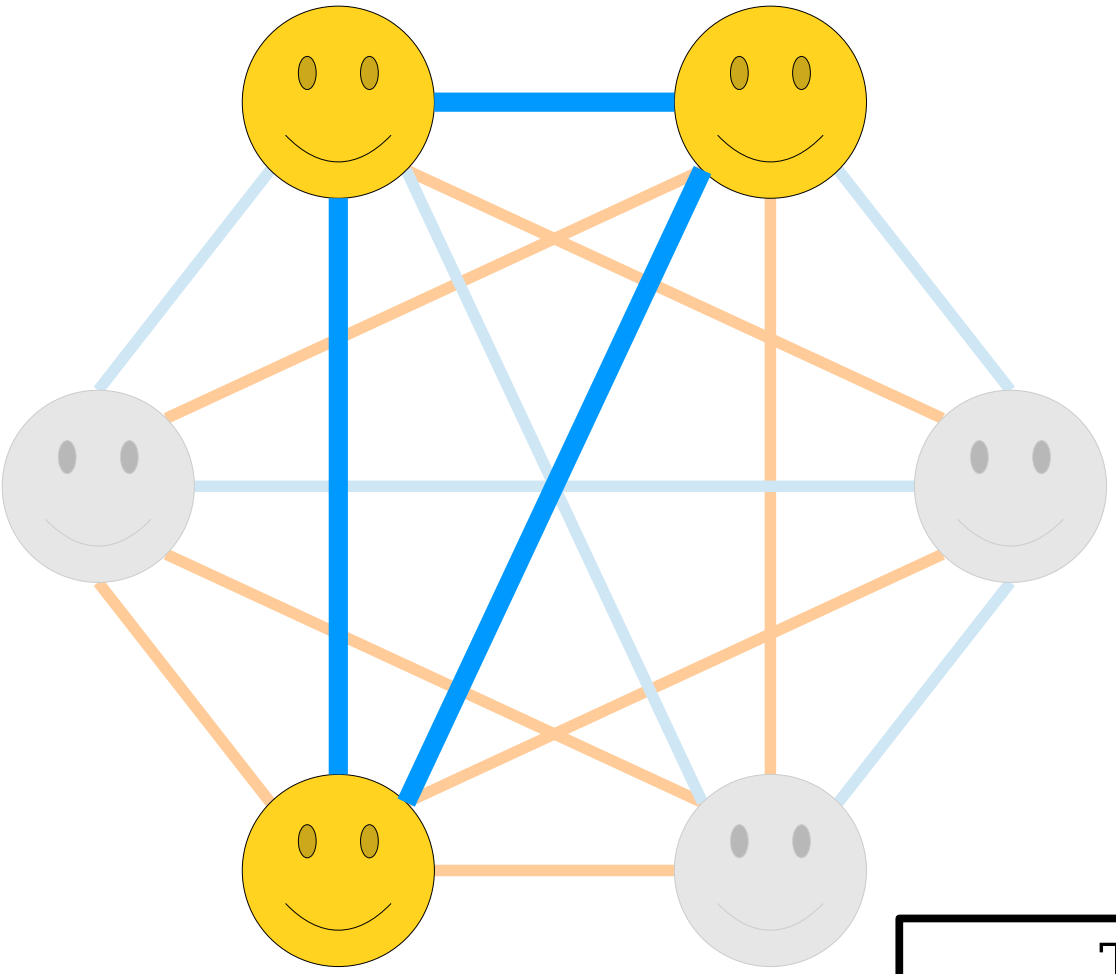




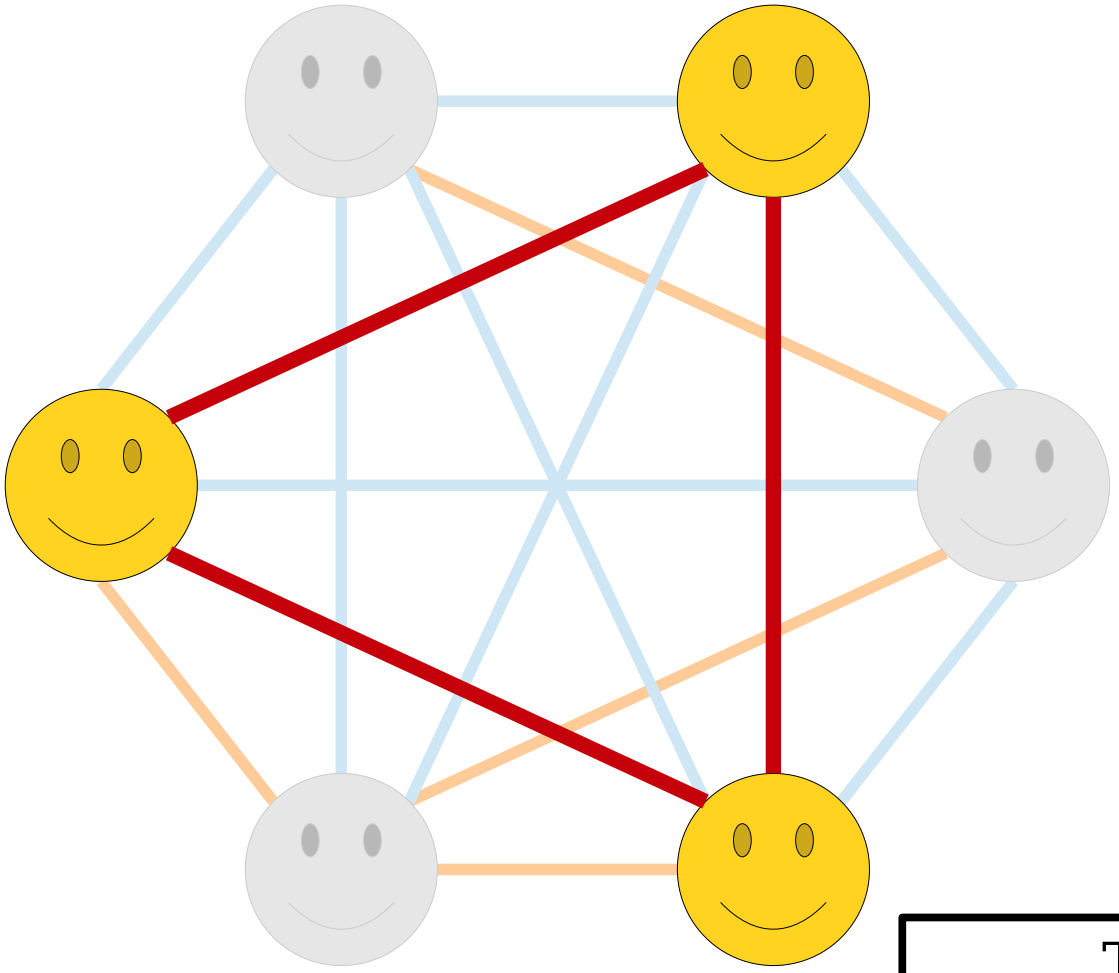
This graph is called ***K_6***, the ***complete graph of order 6***. More generally, the graph K_n consists of n mutually adjacent nodes.







This is a ***monochrome*** (one-color) copy of K_3.



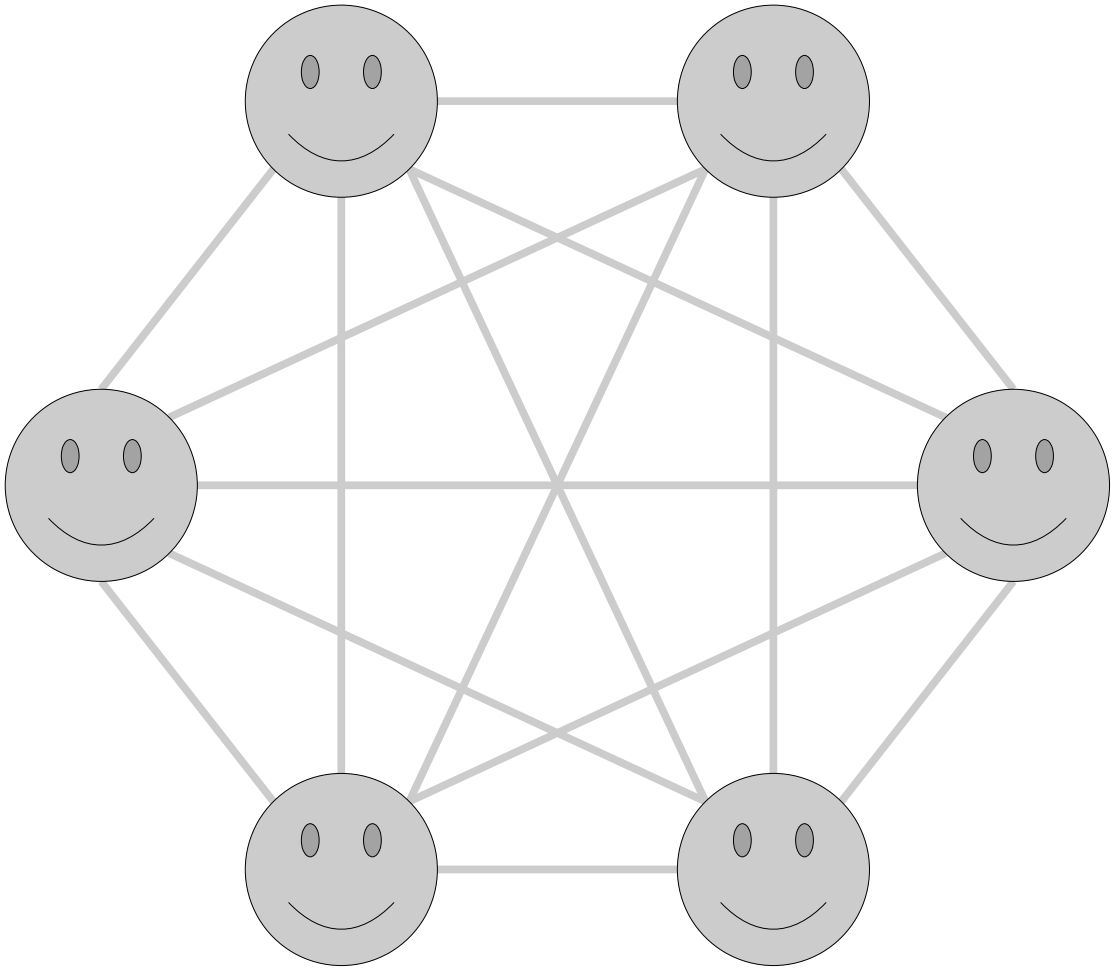
This is a ***monochrome*** (one-color) copy of K_3.

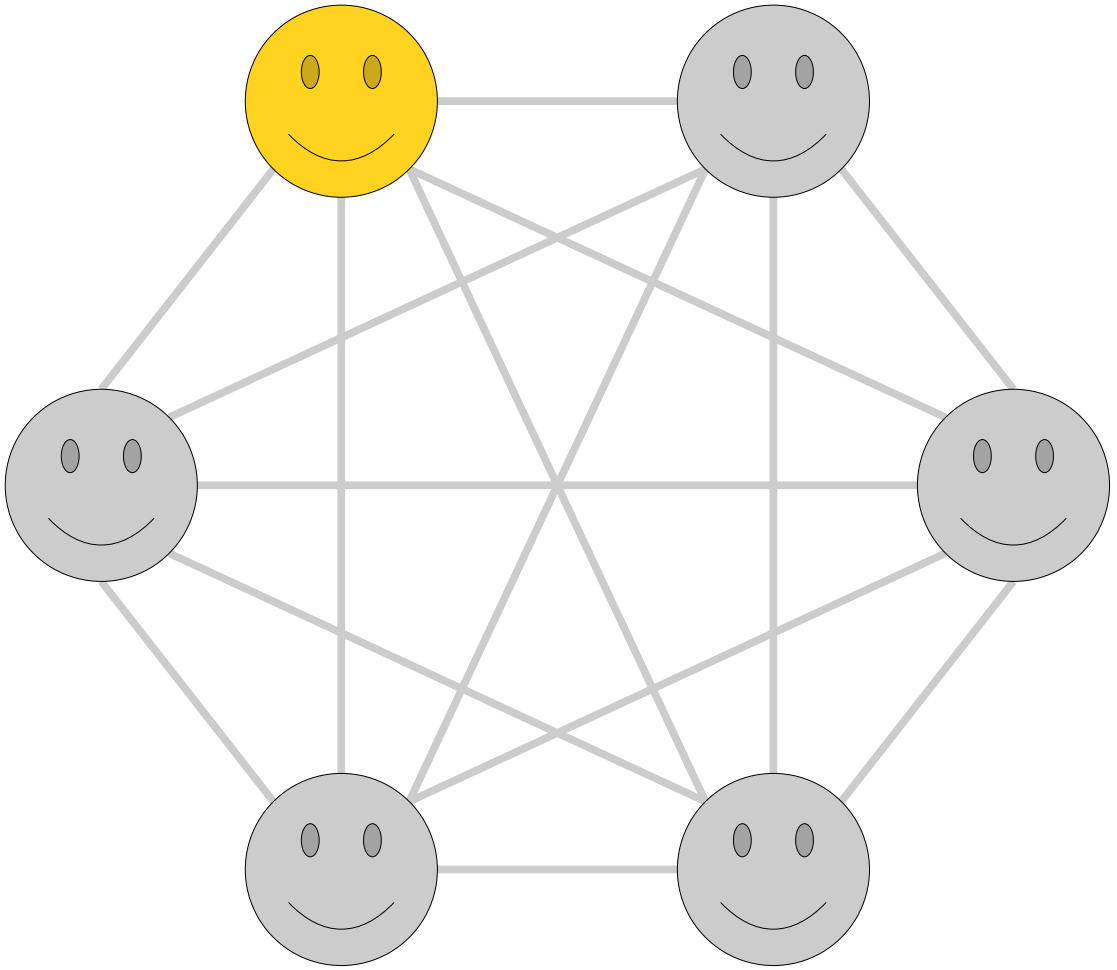
Friends and Strangers Restated

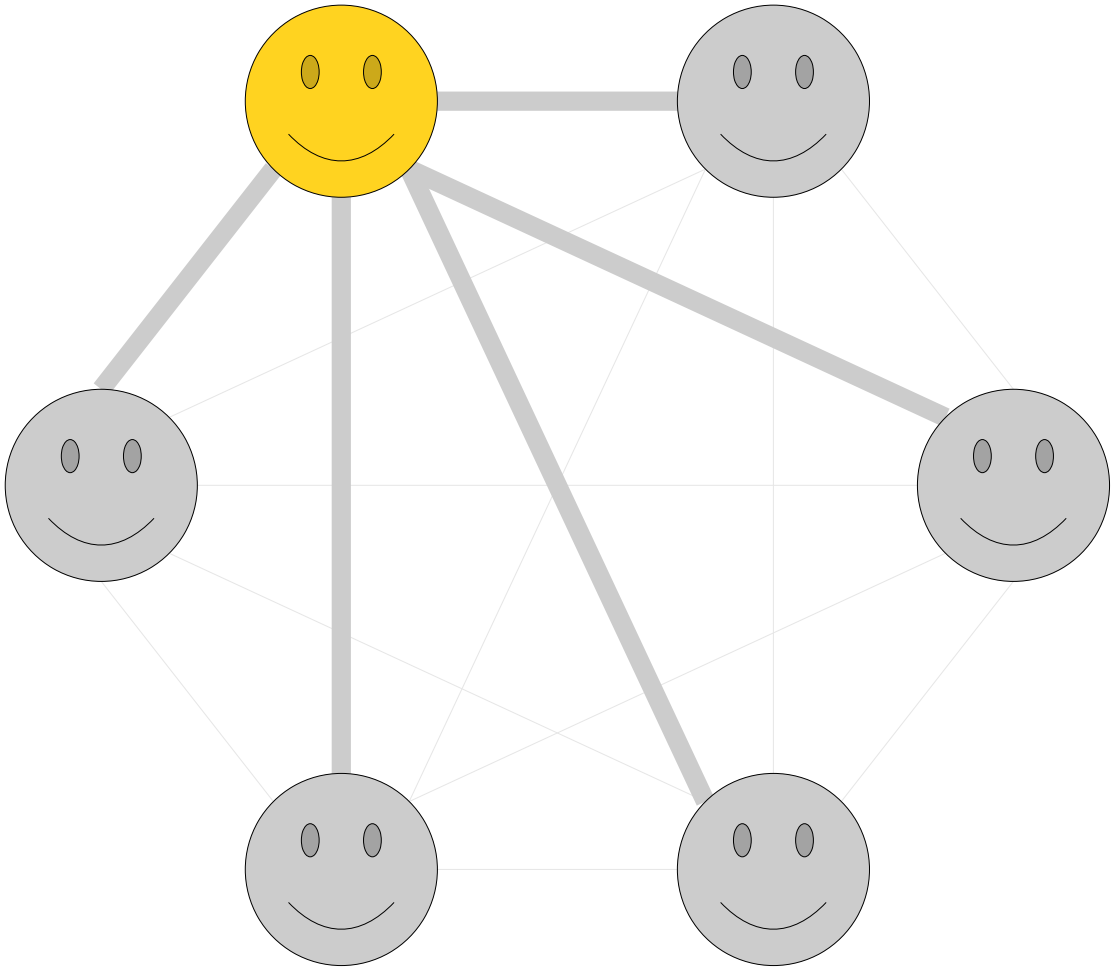
- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

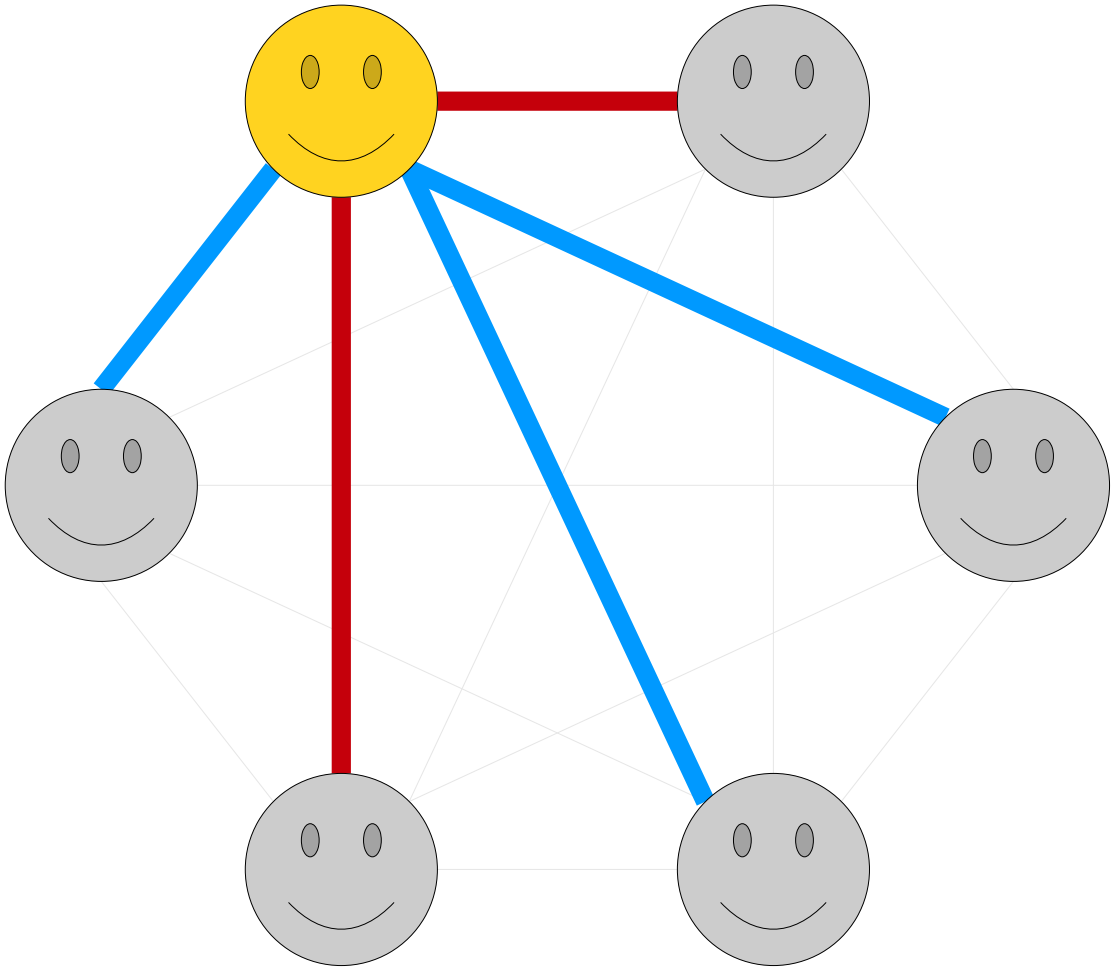
  Theorem: Color every edge of K_6 either red or blue. The resulting graph always contains a monochrome copy of K_3.

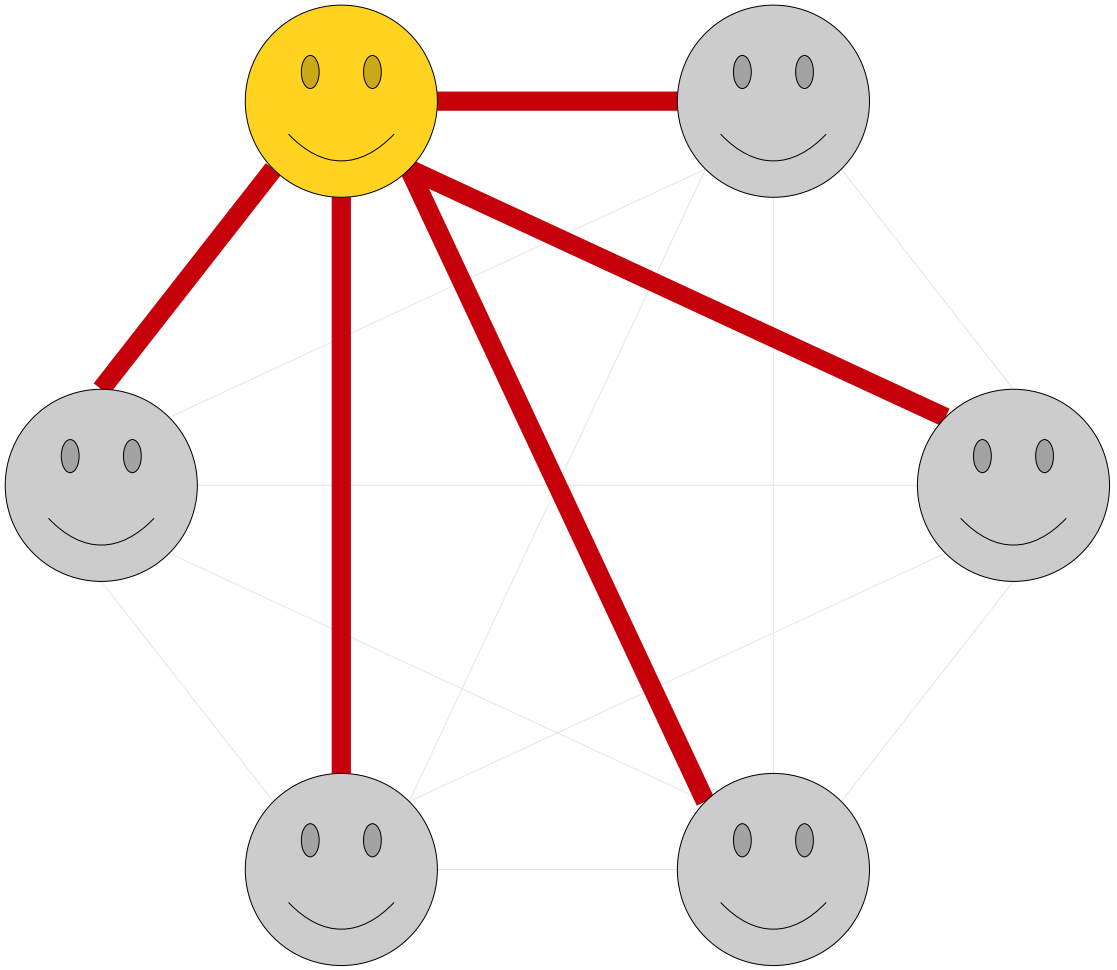
- How can we prove this?

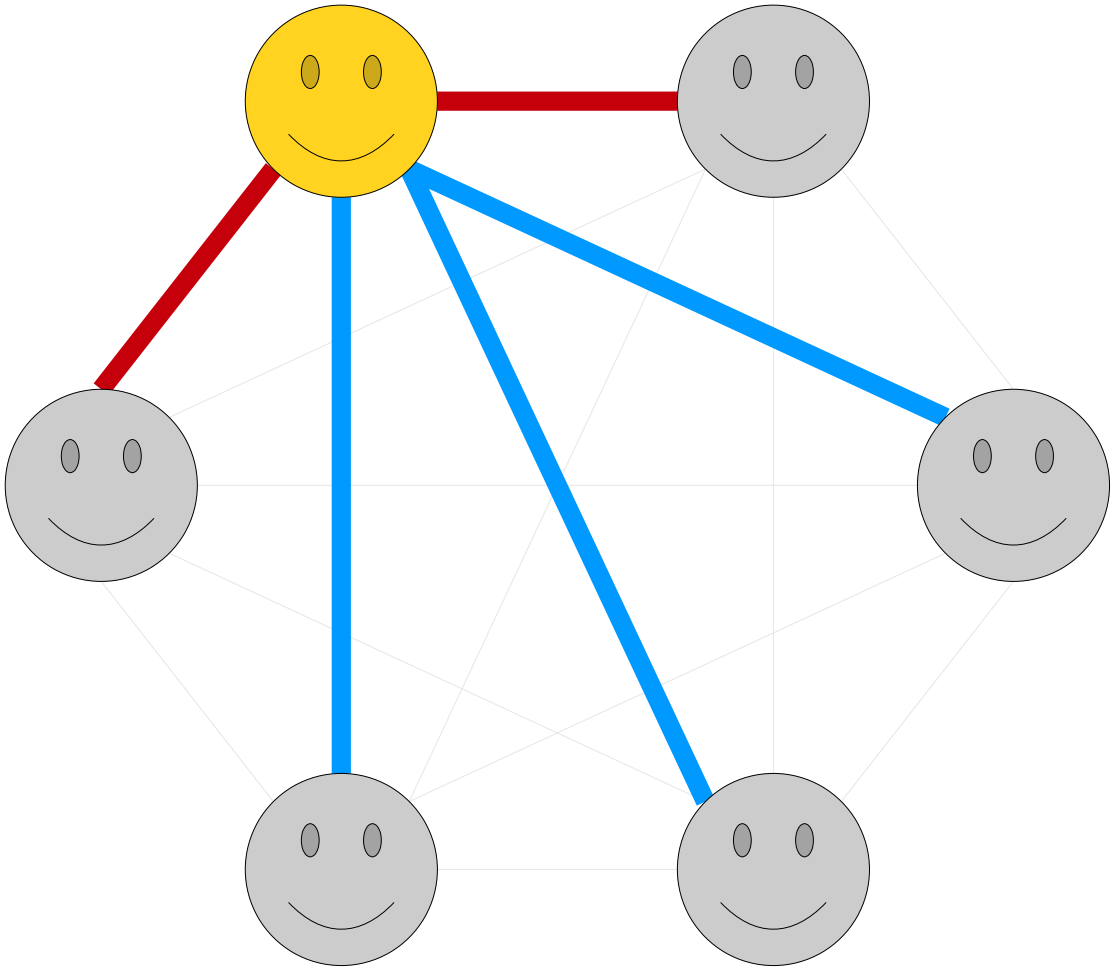


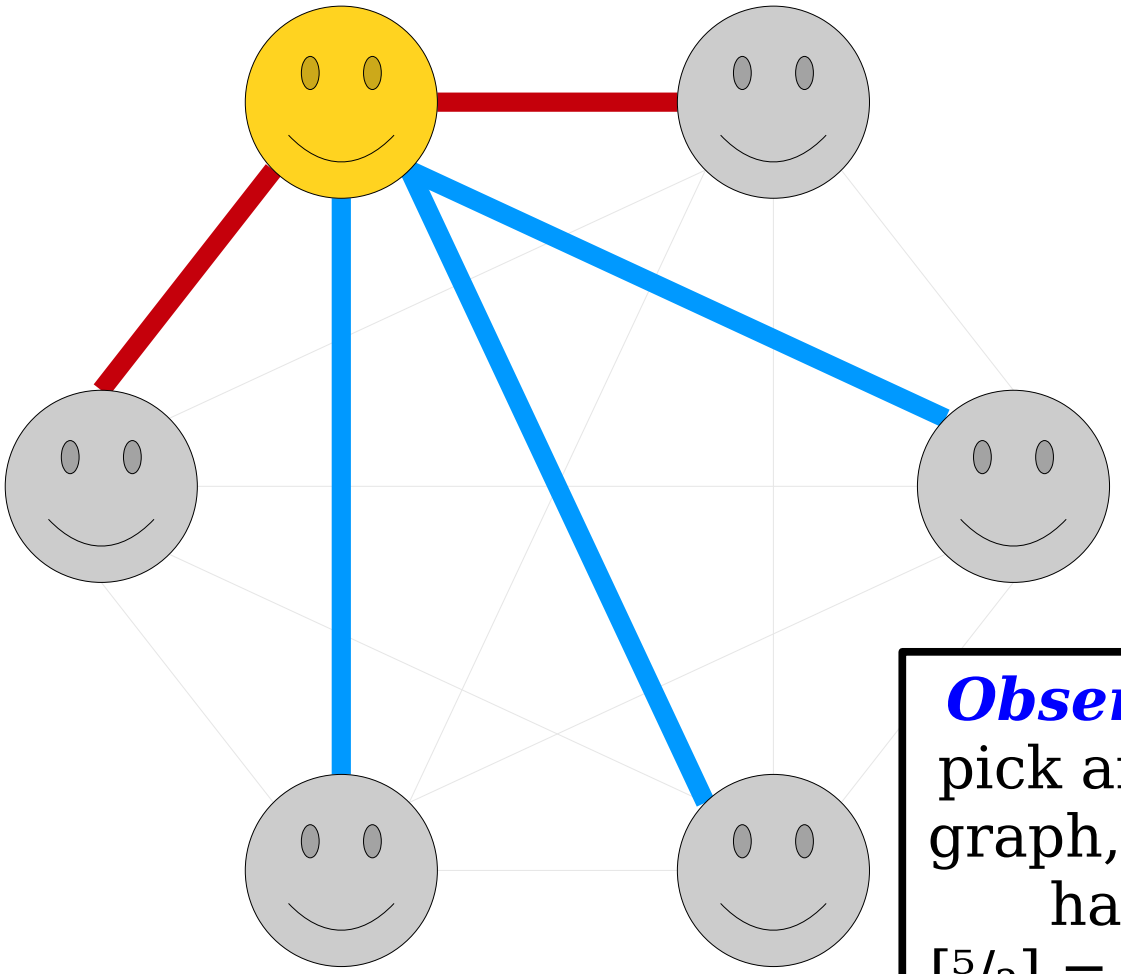




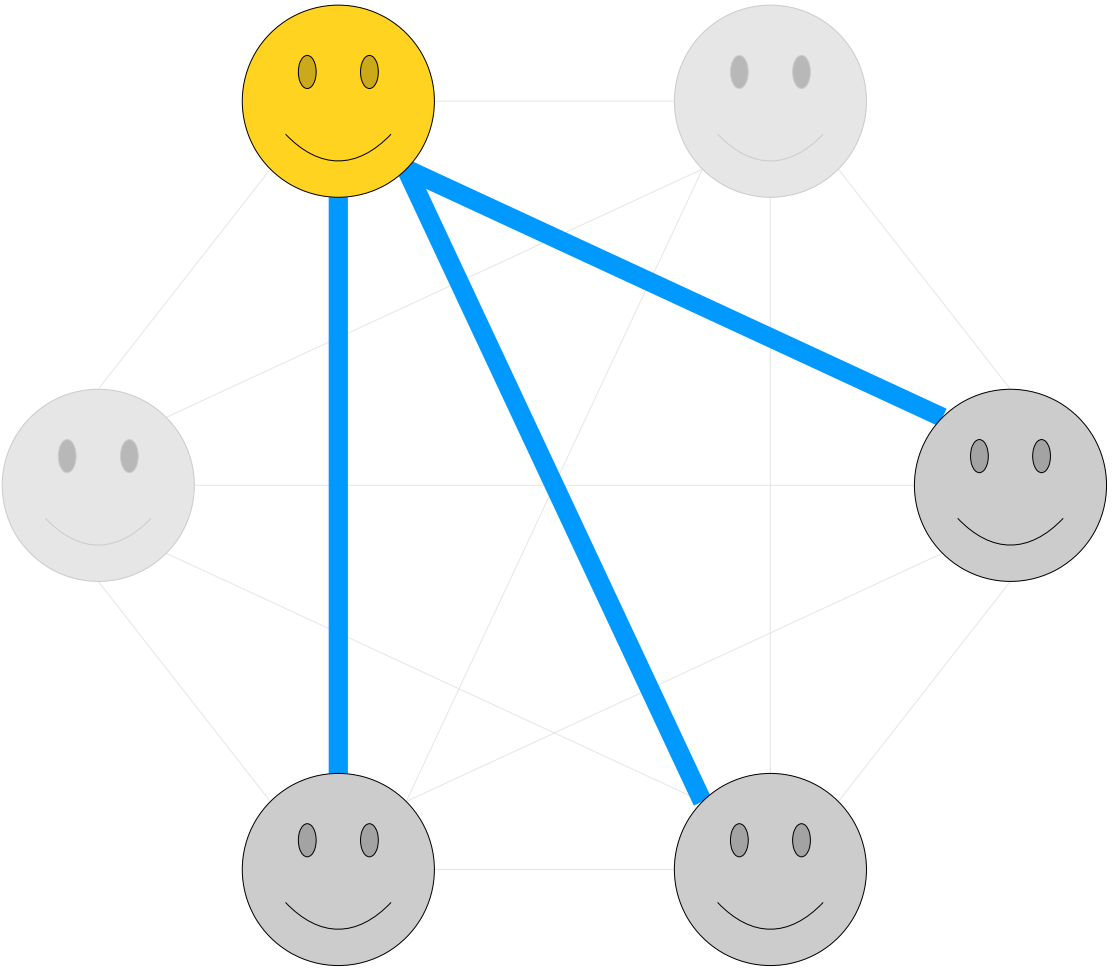


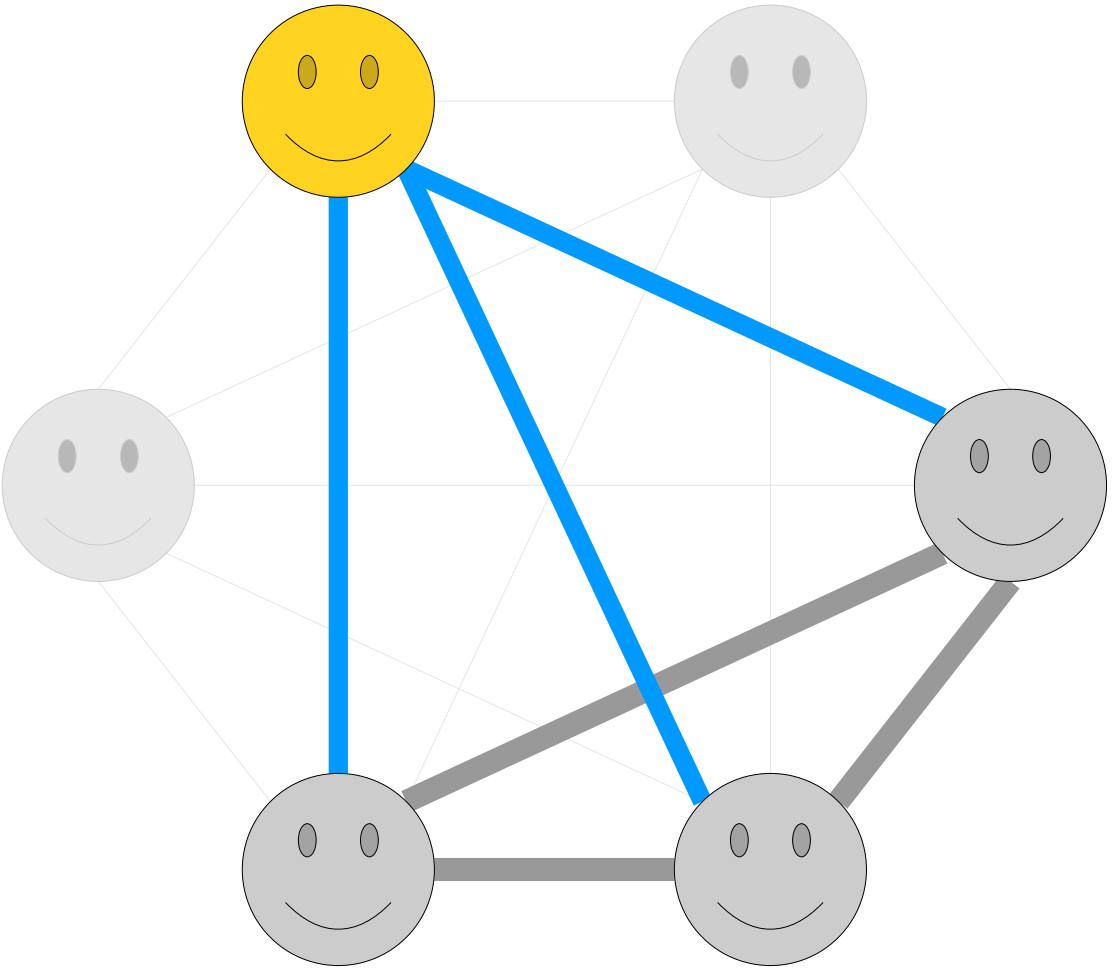


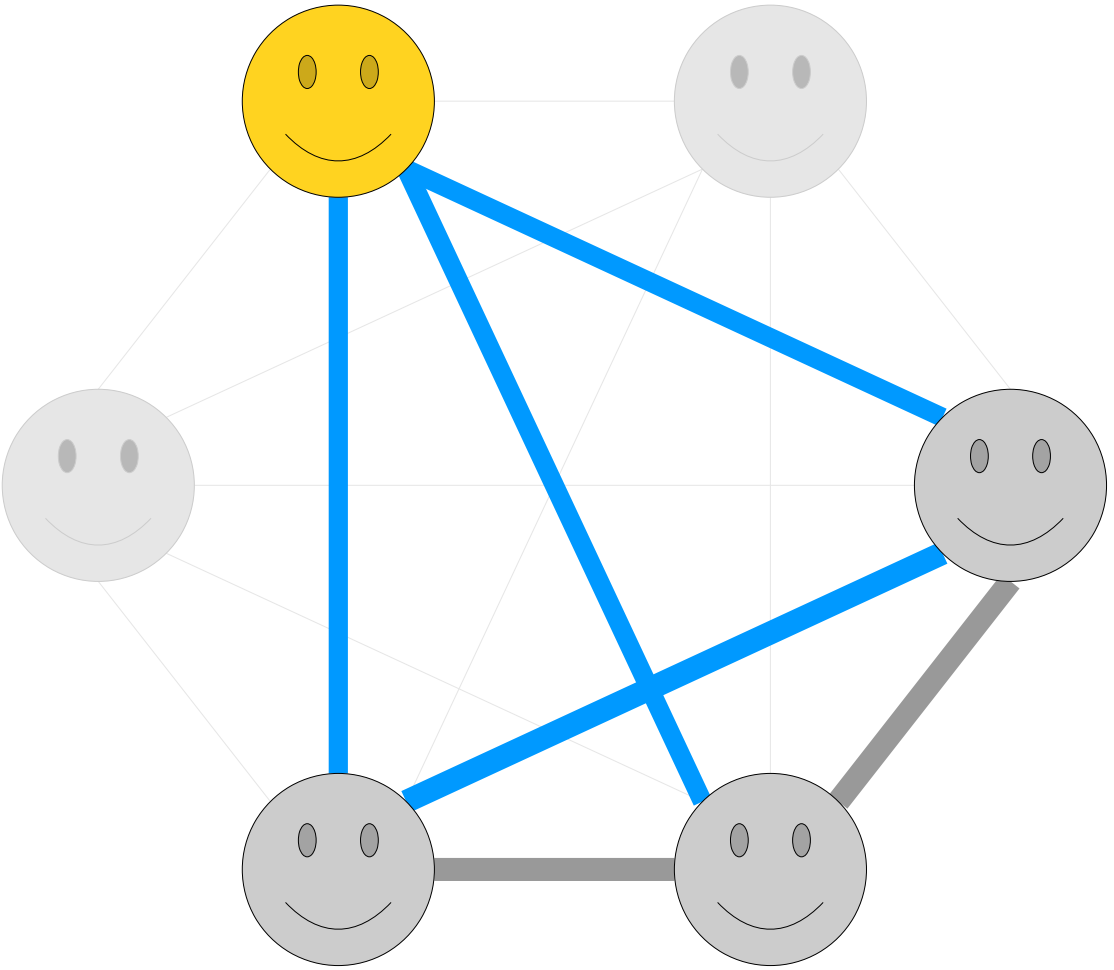


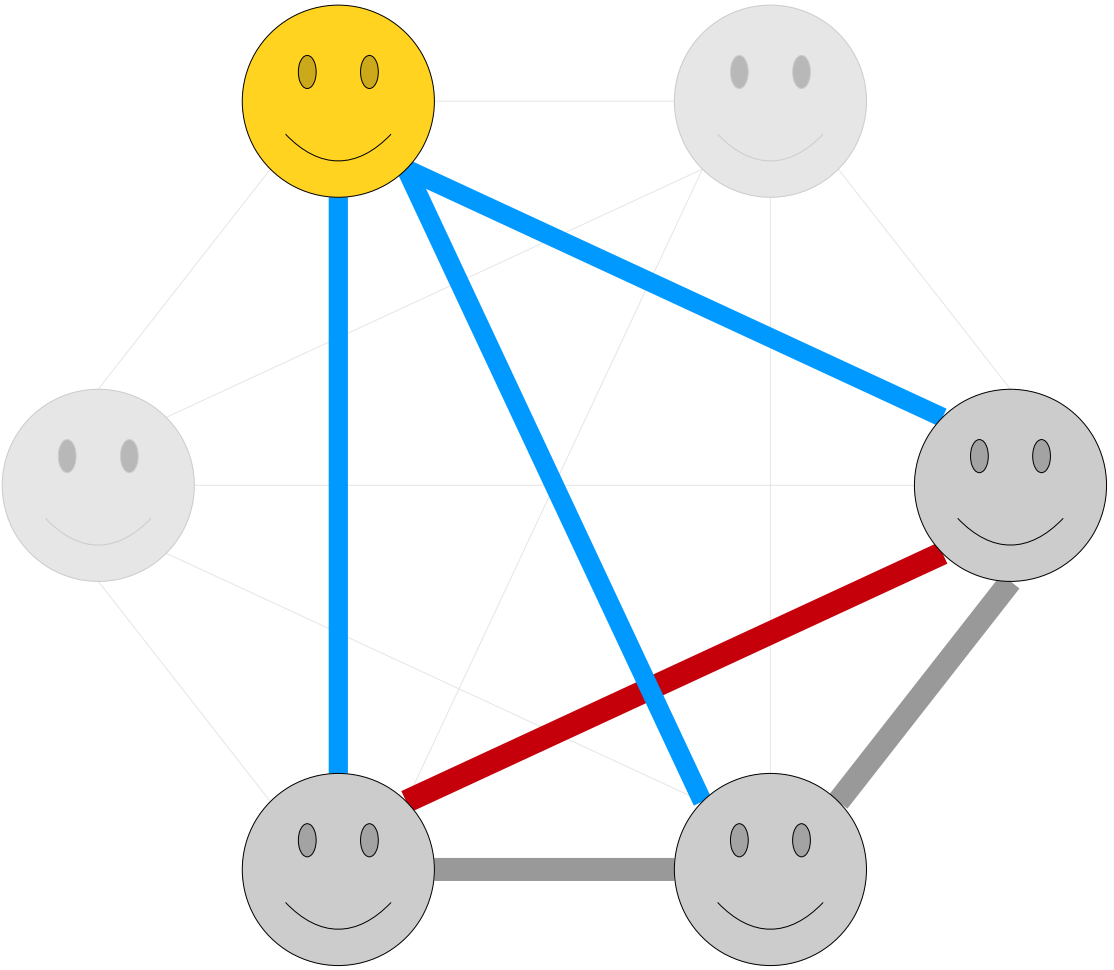


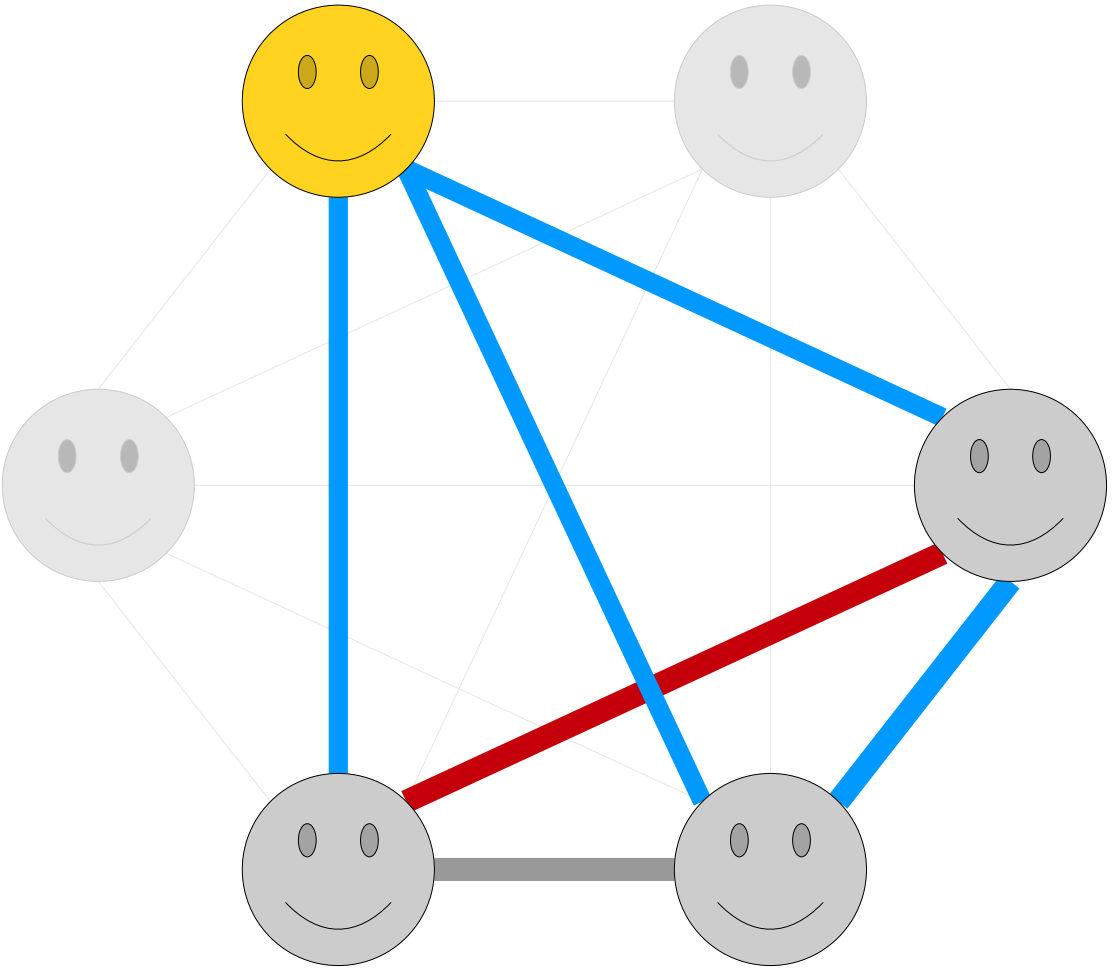
Observation: If we pick any node in the graph, that node will have at least $\lceil 5/2 \rceil = 3$ edges of the same color incident to it.

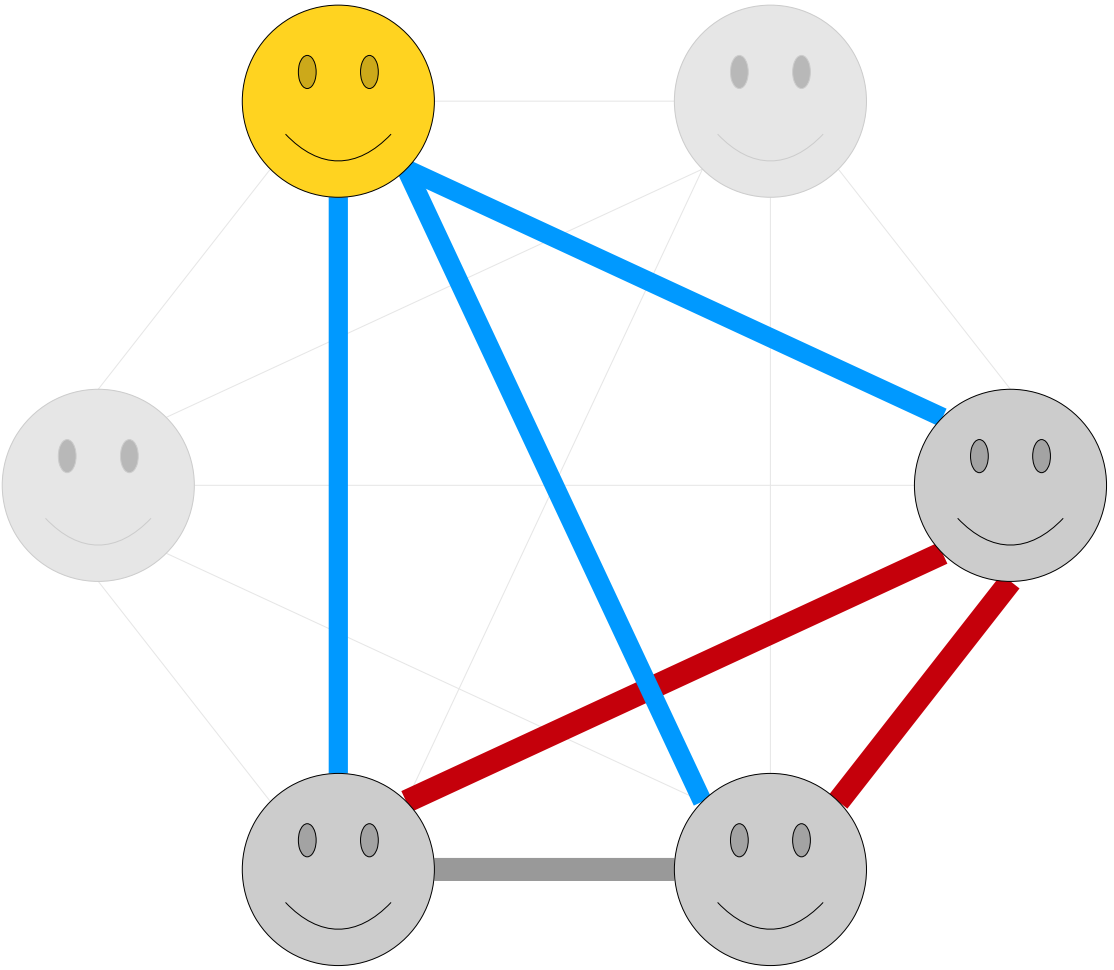


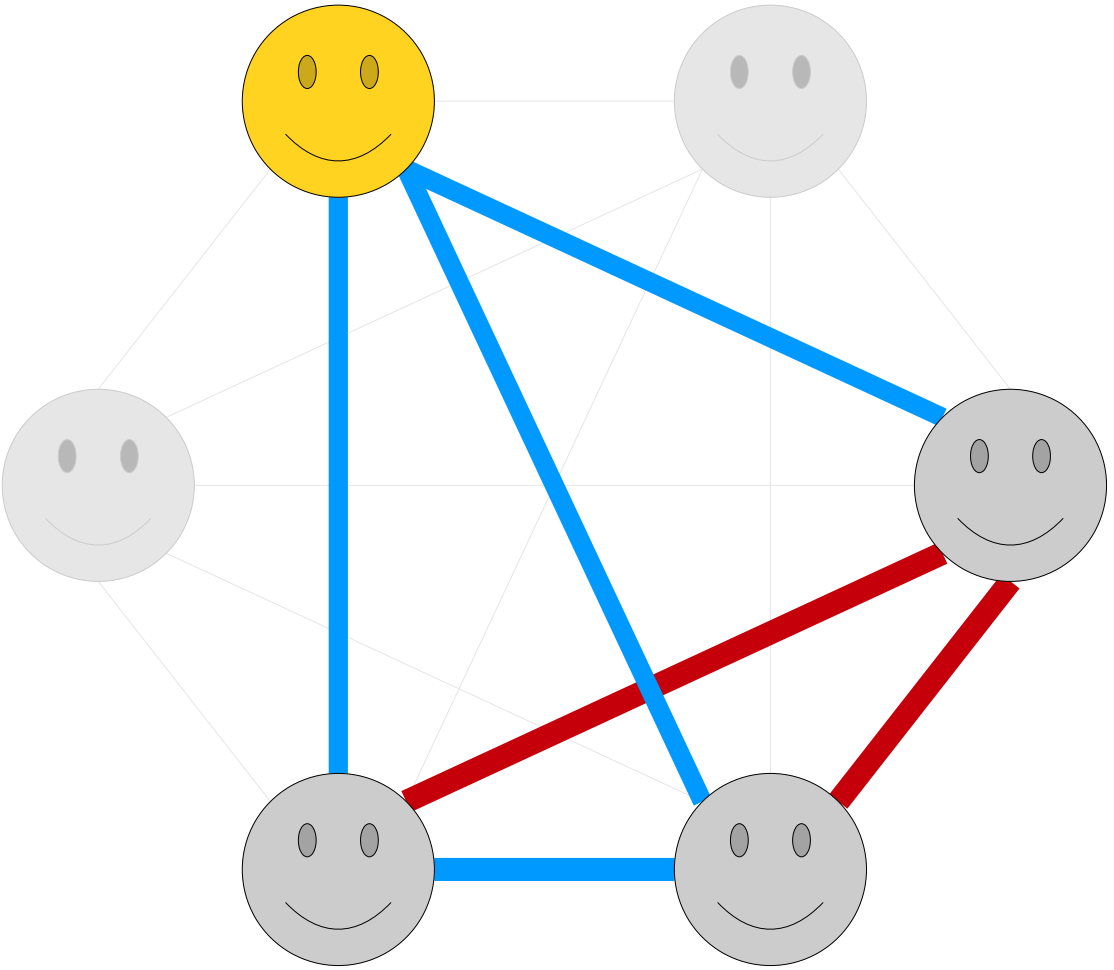


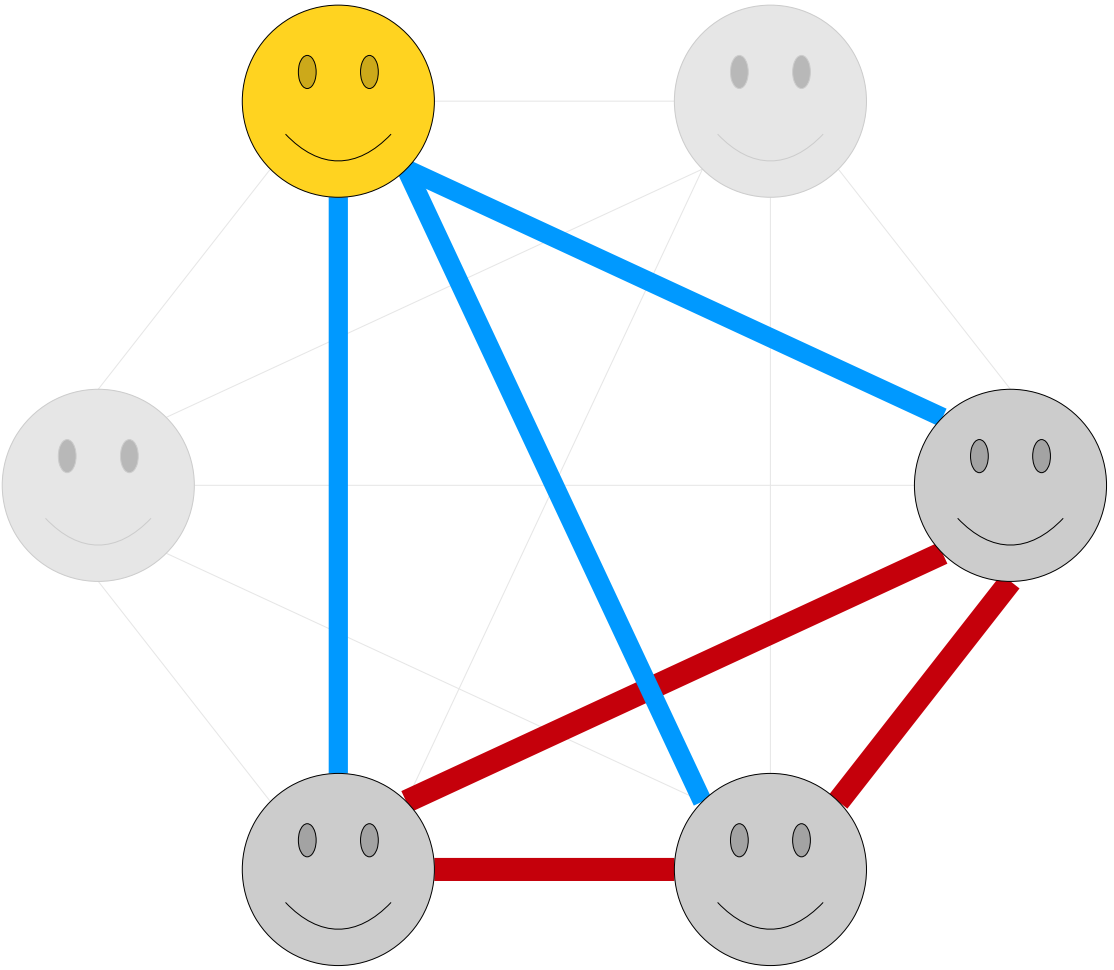












Theorem: Color each edge of K_6 red or blue. The resulting graph contains a monochrome copy of K_3.

Proof: We need to show that the colored K_6 contains a red copy of K_3 or a blue copy of K_3.

Pick some node x from K_6. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Without loss of generality, assume those edges are blue.

Let r, s, and t be three of the nodes adjacent to node x along a blue edge. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are blue, then one of those edges plus the two edges connecting back to node x form a blue K_3. Otherwise, all three of those edges are red, and they form a red K_3. Overall, this gives a red K_3 or a blue K_3, as required. ■

Ramsey Theory

- This proof is a special case of a broader family of results called ***Ramsey theory***.
- ***Theorem (Ramsey):*** For any natural number s, there is a number $R(s)$ such that
  - for all $n < R(s)$, there's a way to color the edges of K_n red and blue so there are no monochrome copies of K_s, and
  - for all $n \geq R(s)$, every way of coloring the edges of K_n red and blue always has a monochrome copy of K_s.
- Take Math 108 (combinatorics) to learn more!
- A more philosophical (and less literal) take on this theorem: true disorder is impossible at a large scale, since no matter how you organize things, you're guaranteed to find some interesting substructure.

The Game of Sim

- Here’s a game you can play with two players.
  - One players plays as red, the other as blue.
  - Begin with six disconnected points.
  - Each turn, a player draws a line of their color.
  - The first to make a triangle of their color loses.
- The theorem we just proved means the game can’t end in a draw: someone must win and someone must lose.
- The strategy is more subtle than it looks. Try playing this with a friend to see why!

Time-Out for Announcements!

Problem Sets

- Problem Set Three was due today at 1:00PM.
  - You can use a late day to extend the deadline to Saturday at 1:00PM if you'd like.
- Problem Set Four due Friday at 1:00PM.
  - It's all about graphs and graph theory, and you'll see some really cool results!
  - Because the midterm is on Monday, we've made this problem set shorter than the previous problem sets.

Participation Opt-Out

- The deadline to opt out of lecture participation and shift the weight to the final exam is ***tonight*** at ***11:59PM***.
- Link is available on EdStem.

Midterm Logistics

- Our first midterm is this upcoming Monday from 7:00PM – 10:00PM.
  - Check the website for seating assignments. ***Write down your seating assignment*** somewhere in case you can't get WiFi signal at the exam.
- Ari is running a review session from 3PM – 4PM in CoDa E160.
- Best of luck on the exam – ***you can do this!*** We're all cheering you on.

Our Advice

- ***Do*** block out some dedicated time to work through practice problems.
- ***Do*** get the TAs to review your answers to those problems; ask privately on Ed.
- ***Do*** take some time this weekend to take a walk, smell the rosemary bushes on campus, and watch the bees buzz.
- ***Don't*** pull an all-nighter studying for the exam.
- ***Don't*** skip meals or alter your daily routine to fit in time for studying.
- ***Don't*** panic. You can do this!

Back to CS103!

A Little Math Puzzle

“In a group of $n > 0$ people …

- 90% of those people enjoyed *CODA*,
- 80% of those people enjoyed *Nomadland*,
- 70% of those people enjoyed *Parasite*, and
- 60% of those people enjoyed *Knives Out*.

No one enjoyed all four movies. How many people enjoyed at least one of *CODA* and *Parasite*?”

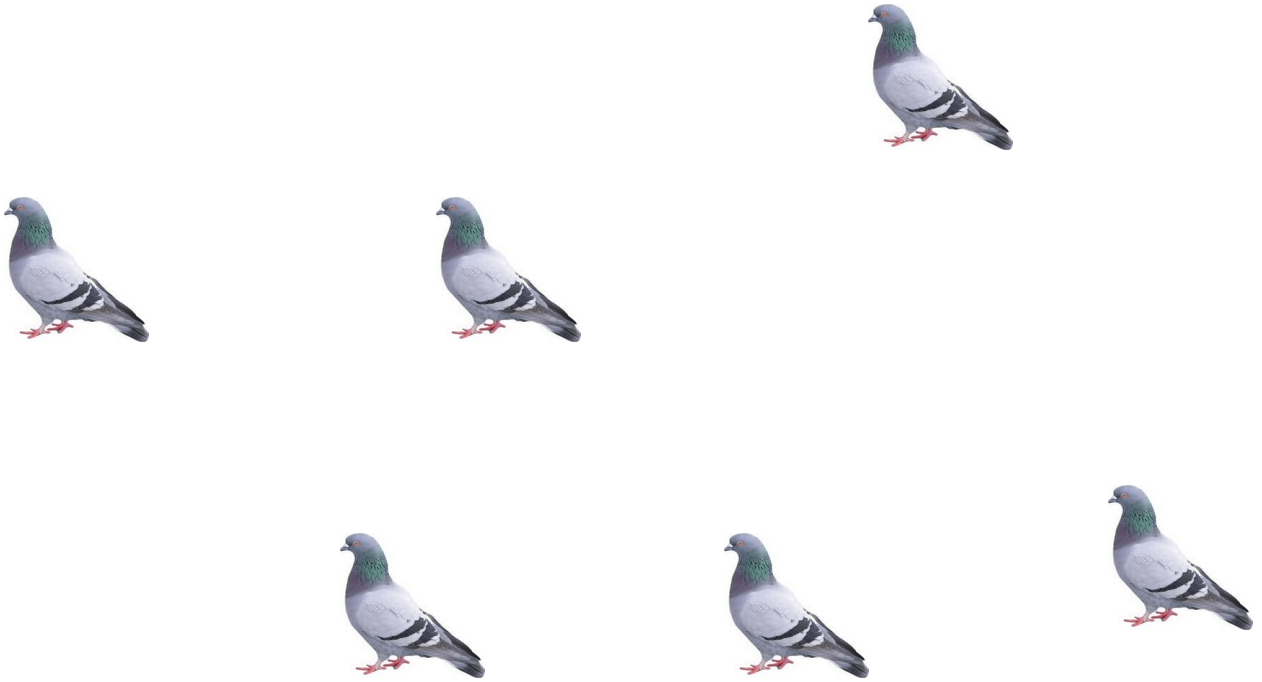
(Adapted from here.)

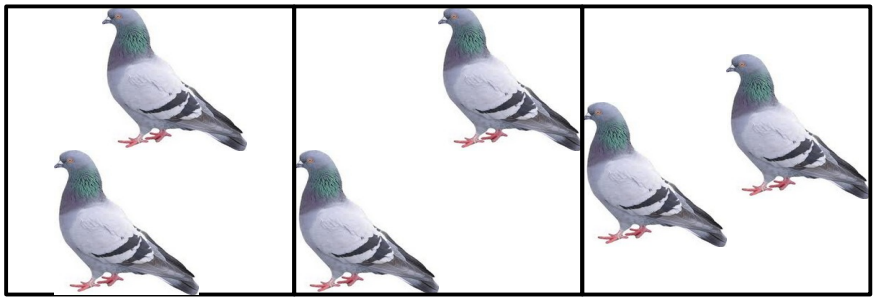
Other Pigeonhole-Type Results

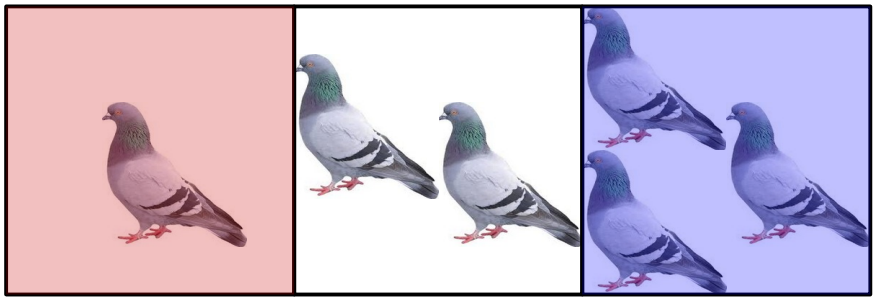
If m objects are distributed into n boxes, then [condition] holds.

If m objects are distributed into n boxes, then some box is loaded to at least the average m/n, and some box is loaded to at most the average m/n.

If m objects are distributed into n boxes, then [condition] holds.







Theorem: If m objects are distributed into n bins, then there is a bin containing more than m/n objects if and only if there is a bin containing fewer than m/n objects.

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

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Proof: Assume for the sake of contradiction that m objects are distributed into n bins such that no bin contains more than m/n objects, yet some bin has fewer than m/n objects.

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For simplicity, denote by x_i the number of objects in bin i.

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For simplicity, denote by x_i the number of objects in bin i. Without loss of generality, assume that bin 1 has fewer than m/n objects, meaning that $x_1 < m/n$.

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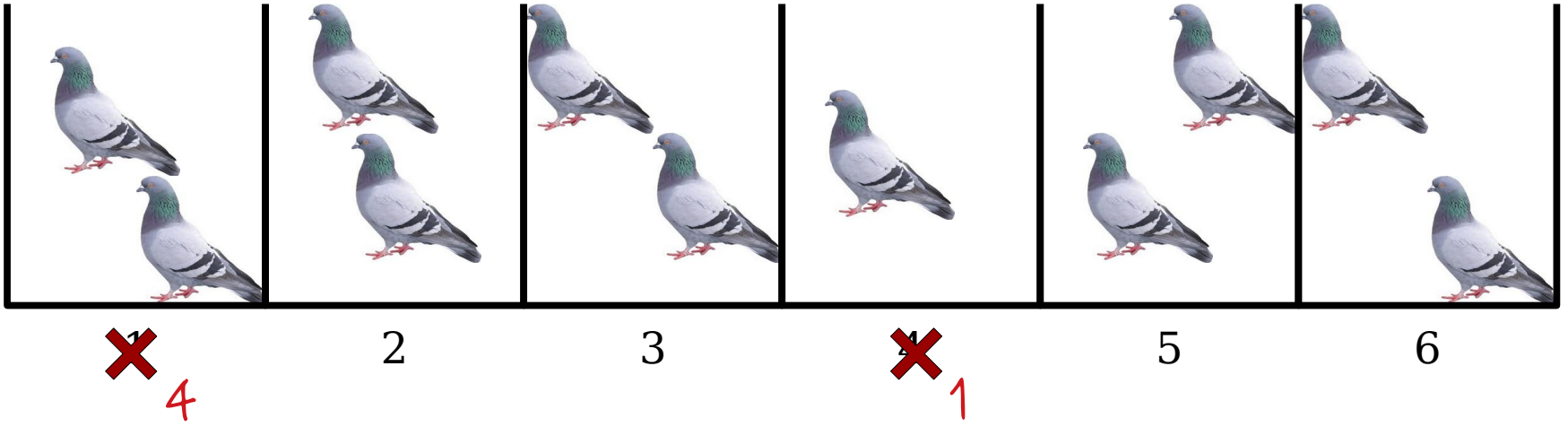
This magic phrase means "we get to pick how we're labeling things anyway, so if it doesn't work out, just relabel things."



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For simplicity, denote by x_i the number of objects in bin i. Without loss of generality, assume that bin 1 has fewer than m/n objects, meaning that $x_1 < m/n$. Adding up the number of objects in each bin tells us that

$$m = x_1 + x_2 + x_3 + \ldots + x_n$$

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$$\begin{aligned} m &= x_1 + x_2 + x_3 + \ldots + x_n \\ &< m/n + x_2 + x_3 + \ldots + x_n \end{aligned}$$

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$$\begin{aligned} m &= x_1 + x_2 + x_3 + \ldots + x_n \\ &< m/n + x_2 + x_3 + \ldots + x_n \\ &\leq m/n + m/n + m/n + \ldots + m/n. \end{aligned}$$

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This third step follows because each remaining bin has at most m/n objects.

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This third step follows because each remaining bin has at most m/n objects. Grouping the n copies of the m/n term here tells us that

$$\begin{aligned} m &< m/n + m/n + m/n + \ldots + m/n \\ &= m. \end{aligned}$$

Lemma: If m objects are distributed into n bins and there are no bins containing more than m/n objects, then there are no bins containing fewer than m/n objects.

Proof: Assume for the sake of contradiction that m objects are distributed into n bins such that no bin contains more than m/n objects, yet some bin has fewer than m/n objects.

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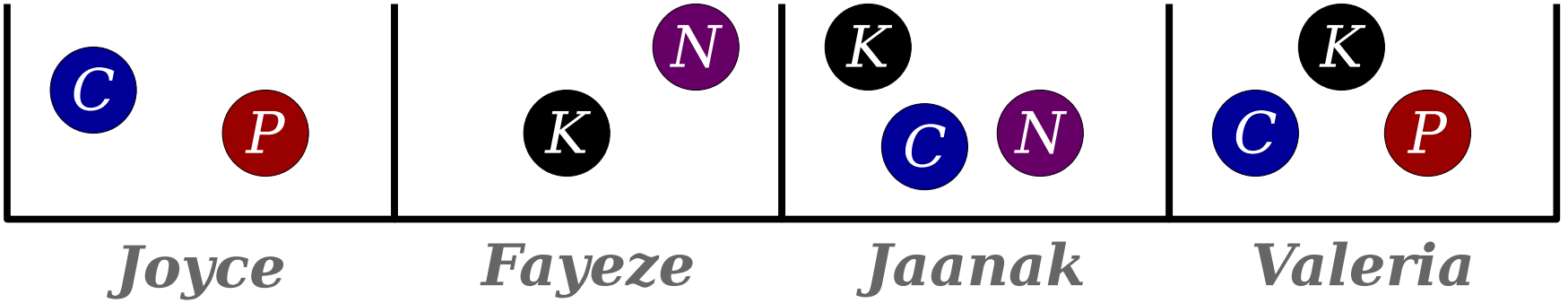
"In a group of $n > 0$ people ...

- 90% of those people enjoyed ***CODA***,
- 80% of those people enjoyed ***Nomadland***,
- 70% of those people enjoyed ***Parasite***, and
- 60% of those people enjoyed ***Knives Out***.

No one enjoyed all four movies. How many people enjoyed at least one of *CODA* and *Parasite*?"

Insight 1: Model movie preferences as balls (movies) in bins (people).

Insight 2: There are n total bins, one for each person.



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$$.9n + .8n + .7n + .6n$$
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Insight 3: There are $3n$ balls being distributed into n bins.

Insight 4: The average number of balls in each bin is 3.

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Insight 5: No one enjoyed more than three movies…

Insight 6: … so no one enjoyed fewer than three movies …

Insight 7: … so everyone enjoyed exactly three movies.

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Insight 8: You have to enjoy at least one of these movies to enjoy three of the four movies.

Conclusion: Everyone liked at least one of these two movies!

Theorem: In the scenario described here, all n people enjoyed at least one of *CODA* and *Parasite*.

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Proof: Suppose there is a group of n people meeting these criteria. We can model this problem by representing each person as a bin and each time a person enjoys a movie as a ball.

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$$.9n + .8n + .7n + .6n = 3n,$$

and since there are n people, there are n bins.

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Going Further

- The pigeonhole principle can be used to prove a *ton* of amazing theorems. Here's a sampler:
  - There is always a way to fairly split rent among multiple people, even if different people want different rooms. *(Sperner's lemma)*
  - You and a friend can climb any mountain from two different starting points so that the two of you maintain the same altitude at each point in time. *(Mountain-climbing theorem)*
  - If you model coffee in a cup as a collection of infinitely many points and then stir the coffee, some point is always where it initially started. *(Brower's fixed-point theorem)*
  - A complex process that doesn't parallelize well must contain a large serial subprocess. *(Mirksy's theorem)*
  - Any positive integer *n* has a nonzero multiple that can be written purely using the digits 1 and 0. *(Doesn't have a name, but still cool!)*

More to Explore

- Interested in more about graphs and the pigeonhole principle? Check out…
  - … ***Math 107*** (Graph Theory), a deep dive into graph theory.
  - … ***Math 108*** (Combinatorics), which explores a bunch of results pertaining to graphs and counting things.
  - … ***CS161*** (Algorithms), which explores algorithms for computing important properties of graphs.
  - … ***CS224W*** (Deep Learning on Graphs), which uses a mix of mathematical and statistical techniques to explore graphs.
- Happy to chat about this in person if you'd like.

Next Time

- ***No Class Monday (Midterm)***
  - Good luck on the midterm!
- ***Then, when we get back…***
  - ***Mathematical Induction***
    - Reasoning about stepwise processes!
  - ***Applications of Induction***
    - To numbers!
    - To anticounterfeiting!
    - To modern art!